

INTEGRATION ON LOCALLY COMPACT NONCOMMUTATIVE SPACES

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ABSTRACT. We present an ab initio approach to integration theory for nonunital spectral triples. This is done without reference to local units and in the full generality of semifinite noncommutative geometry. The main result is an equality between the Dixmier trace and generalised residue of the zeta function and heat kernel of suitable operators. We also examine definitions for integrable bounded elements of a spectral triple based on zeta function, heat kernel and Dixmier trace techniques. We show that zeta functions and heat kernels yield equivalent notions of integrability, which imply Dixmier traceability.

1. INTRODUCTION

The definition of nonunital spectral triples is dictated by Kasparov theory, however the correct method of specifying summability has not been established on a firm base. In particular we need a better understanding of the relationship between the residues of the zeta function, asymptotics of the heat kernel and their relationship with the Dixmier trace. Previous treatments of this question rely on a quasi-localness assumption that allows one to reduce to a ‘compactly supported case’, a notion which may or may not be generally applicable, [18, 32, 33]. Here we seek a more general point of view that avoids local units. Moreover we treat the problem in the general setting of semifinite noncommutative geometry.

It is known by Connes’ trace theorem, [11], that for a *compact* Riemannian manifold M of dimension p , there is an intimate connection between the residue of the zeta function of the Laplacian at its first singularity, the Dixmier trace, and integration theory of functions with respect to the standard measure. More precisely, we let Δ be a Laplace type operator acting on sections of a vector bundle and $f \in C^\infty(M)$ be a smooth function acting by multiplication on smooth sections. Then Δ extends to an essentially self-adjoint operator with compact resolvent on L^2 -sections and the multiplication operator M_f extends to a bounded operator. Then if $G := (1 + \Delta)^{-p/2}$, the operator $M_f G$ has singular numbers $\mu_n = O(\frac{1}{n})$, and the Dixmier trace of this operator is equal to (up to a constant depending only on the dimension p) the integral of f over M with respect to the measure in M given by the volume form associated to a choice of a Riemannian metric. The residue of the zeta function $\text{Tr}(M_f G^s)$ at $s = 1$ and the rescaled limit of $\text{Tr}(M_f e^{-t\Delta})$ when $t \rightarrow 0^+$, also coincide (up to a constant) with the integral of f .

More recently in [22, 23] it has been shown that for $f \in L^1(M)$ we can define the zeta function $\text{Tr}(G^{s/2} M_f G^{s/2})$ for $s > 1$ and that the (generalised or ω) residue at $s = 1$ is equal to the integral

of f over M , while the Dixmier trace of $G^{1/2}M_fG^{1/2}$ does not exist for all f in $L^1(M)$. Related results which indicate the difficulties of noncommutative integration are that the Dixmier trace of M_fG exists if and only if $f \in L^2(M)$. In summary, the generalised residue of the zeta function exists in greater generality than the Dixmier trace, and recovers integration on the manifold. When the Dixmier trace also exists, it agrees with the zeta residue (up to a constant). These results clearly indicate the subtleties one has to face when manipulating products, symmetrised or unsymmetrised, even in the compact manifold case.

When the manifold is not compact the situation is much less clear. Moreover the analogous noncommutative integration theory (for nonunital pre- C^* -algebras) has not been developed from first principles, rather, an assumption is made about the existence of a system of local units for the algebra, [32, 33]. This is a plausible assumption given that there is a local structure on a noncompact manifold M . Moreover this assumption allows one to show that various possible definitions agree, and has been used to prove results analogous to the unital case, such as relating the zeta function and Dixmier trace in that setting, [18, 32, 33]. Noncommutative examples which fit into this framework are described in [18, 27, 28, 32].

Thus we ask what can we learn about noncommutative integration by beginning with the zeta function, rather than the Dixmier trace, in the nonunital case. More precisely we phrase this, our fundamental question, as follows: *what is the appropriate noncommutative integral for nonunital spectral triples?*

There are various possible answers to our fundamental question and we aim, in this note, to explain one, which in view of the results we obtain, is potentially the most natural. A second question, which we believe our approach helps to understand, is *what summability hypotheses are needed to prove the local index formula for nonunital spectral triples*. We shall address this question in another place. Thus for the moment our focus is on whether there are reasonable notions of integration, integrability, and most fundamentally, spectral dimension in the non-unital case. Importantly, we will couch our results, and choose proofs, that go through for the general framework of semifinite spectral triples.

To explain this further we need to resolve a notational issue. Let \mathcal{N} be a semifinite von Neumann algebra with fixed faithful normal semifinite trace τ . In [8] we introduced a family of ideals that we denoted by $\mathcal{Z}_p := \mathcal{Z}_p(\mathcal{N}, \tau)$. These ideals are naturally defined by the asymptotics of the zeta function $s \mapsto \tau(T^s)$ as s converges from above to the infimum of values for which this trace is finite. As a consequence of our analysis we found that \mathcal{Z}_1 coincided with the (dual to the) Macaev ideal for which it has become standard in noncommutative geometry to use the notation $\mathcal{L}^{1,\infty}$. In this paper we will use the notation \mathcal{Z}_p even in the case $p = 1$ for consistency. We make more comments on this matter later.

Now, the problem that arises for noncommutative and nonunital integration is that we are dealing with products of operators, neither of which individually lies in \mathcal{Z}_1 but whose product does lie in \mathcal{Z}_1 . Examples show that a similar phenomenon persists in the case of noncommutative

algebras [18, 27, 28] and general semifinite traces. The difficulties posed by this situation for the analysis of Schrödinger operators has been explained in detail in [36, Chapter $f(X)g(-i\nabla)$].

The aim of this article is to study the general situation from a point of view as close as possible to that of the unital case without assuming the existence of a quasi-local structure.

We begin with some preliminary facts in Sections 2 and 3. One of our first results in Section 4 is to establish that we could just as well study the heat kernel asymptotics. That is, the heat kernel asymptotics imply precisely the same data and exist in the same generality as the generalised zeta residue. We next show that for bounded operators $a \in \mathcal{N}$, the existence of the generalised ζ -residue $\omega - \lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr}(a^* G^{1+\frac{1}{r}} a)$ implies that $a^* G a \in \mathcal{Z}_1$. A range of similar statements shows the compatibility of the zeta residue with the ideals \mathcal{Z}_p and $\mathcal{L}^{p,\infty}$. We use implicitly a Hilbert algebra framework (described in Section 4), that is, our approach yields a noncommutative L^2 theory.

Our main result is proved in Section 5, namely that under ‘suitable conditions’, the zeta function or the heat kernel can be used to compute the Dixmier trace. We seek the weakest hypotheses feasible for this main result with the consequence that the proof is surprisingly subtle and lengthy. We finish in Section 7 by showing that there are definitions of spectral triple and spectral dimension in the nonunital case for which these ‘suitable conditions’ may be verified.

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2. THE IDEALS \mathcal{Z}_p

2.1. Definitions and notations. We fix a semifinite von Neumann algebra \mathcal{N} acting on a separable Hilbert space \mathcal{H} . Fix also a faithful, semifinite, normal trace $\tau : \mathcal{N} \rightarrow \mathbb{C}$. The zeta function of a positive τ -compact operator T is given by $\zeta(s) = \tau(T^s)$ for real positive s , on the assumption that there exists some s_0 for which the trace is finite. Note that it is then true that $\tau(T^s) < \infty$ for all $s \geq s_0$.

Let us consider the space \mathcal{Z}_1 introduced in [8]:

$$\mathcal{Z}_1 := \mathcal{Z}_1(\mathcal{N}, \tau) := \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{Z}_1} = \limsup_{p \searrow 1} (p-1) \tau(|T|^p) < \infty \right\}.$$

Note that $\|\cdot\|_{\mathcal{Z}_1}$ is a seminorm, only. The next equality is easy to see (for details consult [8])

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p \searrow 1} (p-1) \left(\int_0^\infty \mu_t(|T|^p) dt \right)^{1/p} = \limsup_{p \searrow 1} (p-1) \|T\|_p.$$

(We use the notation \mathcal{L}^p for the Schatten ideals in (\mathcal{N}, τ) and $\|\cdot\|_p$ for the Schatten norms.) More generally, we define for $p \geq 1$ the spaces \mathcal{Z}_p , the p -convexifications of \mathcal{Z}_1 [21], by

$$\mathcal{Z}_p(\mathcal{N}, \tau) = \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{Z}_p} = \limsup_{q \searrow p} ((q-p) \tau(|T|^q))^{1/q} < \infty \right\}.$$

An important role in noncommutative geometry is played by the ideal of compact operators whose partial sums of singular values are logarithmically divergent. This ideal (in the setting of general semifinite von Neumann algebras) is probably best expressed through the terminology of noncommutative Marcinkiewicz spaces and we refer to [8, 9, 24] for a detailed exposition of relevant parts of this theory. Here, we set

$$M_{1,\infty}(\mathcal{N}, \tau) := \left\{ T \in \mathcal{N} : \|T\|_{1,\infty} := \sup_{0 < t < \infty} \log(1+t)^{-1} \int_0^t \mu_s(T) ds < \infty \right\}.$$

We will usually take (\mathcal{N}, τ) as fixed, and write $M_{1,\infty}$ instead of $M_{1,\infty}(\mathcal{N}, \tau)$, as this will cause no confusion. Similar comments apply to the notation for other ideals.

The Banach space $(M_{1,\infty}, \|\cdot\|_{1,\infty})$ was probably first considered by Matsaev [26]. It may be viewed as a noncommutative analogue of a Sargent (sequence) space, see [34]. In noncommutative geometry it has become customary to use the notation $\mathcal{L}^{1,\infty}$ to denote the ideal $M_{1,\infty}$. However we will avoid the $\mathcal{L}^{1,\infty}$ notation as it clashes with the well-established notation of quasi-normed weak L_1 -spaces. For a fuller treatment of the history of the space $M_{1,\infty}$ and additional references, we refer the interested reader to the recent paper [29] by Pietsch.

More generally, we let $M_{p,\infty}$, $p \geq 1$, denote the p -convexification of the space $M_{1,\infty}$, defined by

$$(2.1) \quad M_{p,\infty}(\mathcal{N}, \tau) := \left\{ T \in \mathcal{N} : \|T\|_{p,\infty}^p := \sup_{0 < t < \infty} \log(1+t)^{-1} \int_0^t \mu_s(|T|^p) ds < \infty \right\}.$$

In our present context, it is important to observe that it follows from [8, Theorem 4.5] that the sets $M_{1,\infty}$ and \mathcal{Z}_1 coincide and that $\|T\|_0 \leq e\|G\|_{\mathcal{Z}_1}$ and $\|T\|_{\mathcal{Z}_1} \leq \|T\|_{1,\infty}$, where the seminorm $\|\cdot\|_0$ is the distance in the norm $\|\cdot\|_{1,\infty}$ to the subspace $M_{1,\infty}^0$ of $M_{1,\infty}$ formed by the $(1, \infty)$ -norm closure of the trace ideal $\mathcal{L}^1 \subset M_{1,\infty}$. To be consistent with our \mathcal{Z}_p notation, we also denote the latter ideal as \mathcal{Z}_1^0 . Of course, the spaces \mathcal{Z}_p and $M_{p,\infty}$ coincide. We also stress that $\mathcal{L}^{p,\infty}$, $p > 1$, the collection of τ -compact operators for which $\mu_t(T) = O(t^{-1/p})$, are strictly included in \mathcal{Z}_p , [8]. Moreover, a careful inspection of its proof, gives the following strengthening of Theorem 4.5 in [8]:

Theorem 2.1. *The norm $\|\cdot\|_{1,\infty}$ of the Marcinkiewicz space $M_{1,\infty}$ is equivalent to the ζ -norm:*

$$\sup_{p>1} (p-1)\|T\|_p, \quad T \in \mathcal{N}.$$

Another important feature of the ideals $\mathcal{Z}_1 = M_{1,\infty}$ is that they support singular traces. Let $M_{1,\infty}(\mathcal{H})$ denote $M_{1,\infty}$ when (\mathcal{N}, τ) is given by the algebra of all bounded linear operators equipped with standard trace. In [14], J. Dixmier constructed a non-normal semifinite trace living on the ideal $M_{1,\infty}(\mathcal{H})$ using the weight

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{\frac{1}{\log(1+k)} \sum_{j=1}^k \mu_j(T)\right\}_{k=1}^\infty\right), \quad T \geq 0,$$

associated to a translation and dilation invariant state ω on ℓ^∞ . The seminorm $\|\cdot\|_{\mathcal{Z}_1}$ and all Dixmier traces Tr_ω vanish on $M_{1,\infty}^0(\mathcal{H})$ and this provides a first (albeit tenuous) connection between Dixmier traces and zeta functions. This connection runs much deeper however, and will be explained further in various parts of the present manuscript.

2.2. The Calderon-Lozanovskii construction. Let (X, Σ, μ) be a complete σ -finite measure space. By $L_0(X, \mu)$ we will denote the set of all measurable functions which are finite a.e.. As usual we will identify functions equal almost everywhere. A linear subspace of $L_0(X, \mu)$ is called a *Köthe function space* if it is normed space which is an order ideal in $L_0(X, \mu)$, i.e. if $f \in E$ and $|g| \leq |f|$ a.e., then $g \in E$ and $\|g\| \leq \|f\|$. A norm-complete Köthe function space is called a *Banach function space*. Given a Köthe function space E we have that

$$E^\times = \left\{ f \in L_0(X, \mu) : \int_X |fg| d\mu < \infty \text{ for all } g \in E \right\},$$

is a Banach function space with the Fatou property, which is called the associate (or Köthe dual) space of E . Recall that a Banach function space Y has the Fatou property if for any increasing positive sequence (x_n) in Y with $\sup_n \|x_n\|_Y < \infty$ we have that $\sup_n x_n \in Y$ and $\|\sup_n x_n\|_Y = \sup_n \|x_n\|_Y$. This property is equivalent to the apparently stronger property that whenever $\{x_n\}_{n \geq 1} \subseteq Y$, $x \in L_0(X, \mu)$, $x_n \rightarrow x$ almost everywhere and $\sup_n \|x_n\|_Y < \infty$, we have $x \in Y$ and $\|x\|_Y \leq \liminf_{n \rightarrow \infty} \|x_n\|_Y$. For every Banach function space Y we have the embedding $Y \subseteq Y^{\times \times}$ with $\|x\|_{Y^{\times \times}} \leq \|x\|_Y$ for any $x \in Y$. Moreover, $Y = Y^{\times \times}$ with equality of the norms if and only if Y has the Fatou property. For a detailed treatment of Banach function spaces we refer to [20, 21].

We need the Calderon construction of intermediate spaces [6], which was studied and extended by Lozanovskii. Let E and F be Banach function spaces on (X, μ) . Then for $1 < p < \infty$ and $p' = p/(p-1)$, the Banach function space $E^{\frac{1}{p}} F^{\frac{1}{p'}}$ is defined as the space of all $f \in L_0(X, \mu)$ such that $|f| = |g|^{\frac{1}{p}} |h|^{\frac{1}{p'}}$ for some $g \in E$ and $h \in F$. The norm on $E^{\frac{1}{p}} F^{\frac{1}{p'}}$ is defined by

$$\|f\|_{E^{\frac{1}{p}} F^{\frac{1}{p'}}} = \inf \{ \|g\|_E^{\frac{1}{p}} \|h\|_F^{\frac{1}{p'}} : |f| = |g|^{\frac{1}{p}} |h|^{\frac{1}{p'}} \text{ for some } g \in E, h \in F \}.$$

It is well-known that $E^{\frac{1}{p}} F^{\frac{1}{p'}}$ is again a Banach function space, moreover it has order continuous norm if at least one of E and F has order continuous norm. Also $E^{\frac{1}{p}} F^{\frac{1}{p'}}$ has the Fatou property if both E and F have the Fatou property. The above construction contains as a special case the so-called p -convexification of E by taking $F = L^\infty(X, \mu)$. We will denote this space by $E^{\frac{1}{p}}$, since as sets $E^{\frac{1}{p}} L^\infty(X, \mu)^{\frac{1}{p'}} = E^{\frac{1}{p}}$. Note that the norm on $E^{\frac{1}{p}}$ is given by $\|f\|_{E^{\frac{1}{p}}}^p = \| |f|^p \|_E$.

2.3. Complex interpolation. To assist the reader we clarify the construction of the ideals \mathcal{Z}_p in term of complex and real interpolation in the setting of Banach-lattices. For the definition of the two functors of complex interpolation $\bar{A}_{[\theta]}$ (the first method) and $\bar{A}^{[\theta]}$ (the second method) defined for an arbitrary Banach couple $A = (A_0, A_1)$ we refer to [2, pp. 88-90], [25]. In general,

the two spaces $\bar{A}_{[\theta]}$ and $\bar{A}^{[\theta]}$ are not equal, but always $\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}$ with a norm one injection. Usually, the main interest is attached to the space $\bar{A}_{[\theta]}$ and the space $\bar{A}^{[\theta]}$ is more or less only a technical tool. However, sometimes we also need the second method as well. The next result follows immediately from Theorem 1 in [25].

Theorem 2.2. *If the Banach spaces, A_0 and A_1 , have the Fatou property then the closed unit ball in $[A_0, A_1]^{[s]}$ coincides with the closure in $A_0 + A_1$ of the closed unit ball of $A_0^{1-s} A_1^s$.*

Combining this result with the fact that the unit ball of the space $A_0^{1-s} A_1^s$ is closed with respect to convergence in measure and hence also with respect to the norm convergence in $A_0 + A_1$, we arrive at the equality $[A_0, A_1]^{[s]} = A_0^{1-s} A_1^s$.

2.4. Marcinkiewicz function and sequence spaces. These are the main examples of fully symmetric function and sequence spaces. Let Ω denote the set of concave functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow 0+} \psi(t) = 0$ and $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Write x^* for the decreasing rearrangement of the function x : x^* is the right continuous non-increasing function whose distribution function coincides with that of $|x|$, (see [20, 21]). Then for $\psi \in \Omega$ define the weighted mean function

$$a(x, t) = \frac{1}{\psi(t)} \int_0^t x^*(s) ds \quad t > 0.$$

Denote by $M(\psi)$ the (Marcinkiewicz) space of measurable functions x on $[0, \infty)$ such that

$$(2.2) \quad \|x\|_{M(\psi)} := \sup_{t>0} a(x, t) = \|a(x, \cdot)\|_\infty < \infty.$$

We assume in this paper that $\psi(t) = O(t)$ when $t \rightarrow 0$, which is equivalent to the continuous embedding $M(\psi) \subseteq L^\infty([0, \infty))$. The definition of the Marcinkiewicz sequence space $m(\psi)$ of functions on \mathbb{N} is similar.

Example. Introduce the following functions

$$\psi_1(t) = \begin{cases} t \cdot \log 2, & 0 \leq t \leq 1 \\ \log(1+t), & 1 \leq t < \infty \end{cases} \quad \text{and} \quad \psi_p(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ t^{1-\frac{1}{p}}, & 1 \leq t < \infty \end{cases}, \quad p > 1.$$

The spaces $\mathcal{L}^{1,\infty} = \mathcal{Z}_1$ and $\mathcal{L}^{p,\infty} \subset \mathcal{Z}_p$, $p > 1$, are the Marcinkiewicz spaces $M(\psi_1)$ and $M(\psi_p)$ respectively. Now, Lozaniovskiĭ's result described in Section 2.2 and the fact that Marcinkiewicz spaces possess the Fatou property yields

$$[M(\psi), L^\infty([0, \infty))]^{[s]} = M(\psi)^{1-s}.$$

Thus Calderon's second method of complex interpolation applied to a couple $(M(\psi), L^\infty)$ (on an arbitrary measure space) yields the p -convexification of $M(\psi)$ (with appropriate p).

2.5. Marcinkiewicz operator spaces. For a definition and discussion of fully symmetric operator spaces and their important subclass, operator Marcinkiewicz spaces, we refer e.g. to [8, 9, 24]. For \mathcal{N} a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ , let $M(\psi)(\mathcal{N}, \tau)$ be the corresponding operator Marcinkiewicz space. It follows from the results given in [15, 16] that the space $[M(\psi)(\mathcal{N}, \tau), \mathcal{N}]^{[s]}$ coincides with the (fully symmetric) operator space generated by the space $[M(\psi), L^\infty]^{[s]}$ on the von Neumann algebra \mathcal{N} . Taking into account the equality above, we see that $[M(\psi)(\mathcal{N}, \tau), \mathcal{N}]^{[s]} = M(\psi)^{1-s}(\mathcal{N}, \tau)$. It remains to observe that the space $M(\psi)^{1-s}(\mathcal{N}, \tau)$ is precisely the space described in [8] as p -convexification, which is a noncommutative counterpart of the Calderon-Lozanovskii construction outlined earlier. Thus, applying all of the above to the couple $(\mathcal{Z}_1, \mathcal{N})$, we obtain the equality $[\mathcal{Z}_1, \mathcal{N}]^{[1-1/p]} = \mathcal{Z}_p$.

3. THE HEAT KERNEL DEFINITION OF \mathcal{Z}_1

Later we will focus on the zeta function definition of the ideal \mathcal{Z}_1 , but here consider briefly the heat kernel definition. The reason for doing this is to highlight the similarities and differences between \mathcal{Z}_1 and the weak \mathcal{L}^1 ideal.

The weak \mathcal{L}^1 ideal, denoted $\mathcal{L}_{1,w}$, is given by

$$(3.1) \quad \mathcal{L}_{1,w} := \mathcal{L}_{1,w}(\mathcal{N}, \tau) := \{T \in \mathcal{N} : \exists C > 0 \text{ such that } \mu_t(T) \leq C/t\}.$$

A quasi-norm on $\mathcal{L}_{1,w}$ is given by $\|T\|_{1,w} := \inf \{C : \mu_t(T) \leq C/t\}$. Clearly $\mathcal{L}_{1,w} \subseteq \mathcal{Z}_1$, and in fact it was shown in [8] that the inclusion is strict.

The (multiplicative) Cesaro mean, M , on \mathbb{R}_+^* , is defined on $f : \mathbb{R}_+^* \rightarrow [0, \infty)$ by

$$(Mf)(\lambda) := \frac{1}{\log(\lambda)} \int_1^\lambda f(t) \frac{dt}{t}.$$

Then, by [8, Lemma 5.1], $T \in \mathcal{Z}_1$ if and only if $\|Mf_T\|_\infty < \infty$ with $f_T(\lambda) := \lambda^{-1}\tau(e^{-\lambda^{-1}|T|^{-1}})$. A natural question is whether f_T can be unbounded while Mf_T is bounded. If so, is it possible that MMf_T is bounded while Mf_T is unbounded (and so on)?

Proposition 3.1. *Let $T \in \mathcal{Z}_1$ and $0 \leq h : \mathbb{R}_+^* \rightarrow [0, \infty)$. Then*

- 1) *if MMh is bounded, so is Mh ;*
- 2) *the function f_T is unbounded and Mf_T is bounded if and only if $T \in \mathcal{Z}_1$ and $T \notin \mathcal{L}_{1,w}$;*
- 3) *the function f_T is bounded if and only if $T \in \mathcal{L}_{1,w}$.*

Proof. Using the definition and integrating by parts we have

$$\begin{aligned} (MMh)(x) &= \frac{1}{\log(x)} \int_1^x \frac{1}{\log(y)} \int_1^y h(\lambda) \frac{d\lambda}{\lambda} \frac{dy}{y} \\ &= \frac{\log(\log(x))}{\log(x)} \int_1^x h(\lambda) \frac{d\lambda}{\lambda} - \frac{1}{\log(x)} \int_1^x \log(\log(\lambda)) h(\lambda) \frac{d\lambda}{\lambda} \\ &= \frac{1}{\log(x)} \int_1^x \log\left(\frac{\log(x)}{\log(\lambda)}\right) h(\lambda) \frac{d\lambda}{\lambda}. \end{aligned}$$

Now consider $(MMh)(x^2)$:

$$\begin{aligned} (MMh)(x^2) &= \frac{1}{2\log(x)} \int_1^{x^2} \log\left(\frac{2\log(x)}{\log(\lambda)}\right) h(\lambda) \frac{d\lambda}{\lambda} \\ &\geq \frac{1}{2\log(x)} \int_1^x \log\left(\frac{2\log(x)}{\log(\lambda)}\right) h(\lambda) \frac{d\lambda}{\lambda} \geq \frac{\log(2)}{2\log(x)} \int_1^x h(\lambda) \frac{d\lambda}{\lambda} = \frac{\log(2)}{2} (Mh)(x). \end{aligned}$$

Hence $(MMh)(x^2) \geq \frac{\log(2)}{2} (Mh)(x)$ and if MMh is bounded, so too is Mh proving part 1).

For parts 2) and 3), since we know that $T \in \mathcal{Z}_1$ if and only if Mf_T is bounded, it suffices to show that $T \in \mathcal{L}_{1,w}$ if and only if f_T is bounded. So suppose that $T \in \mathcal{L}_{1,w}$ with $\|T\|_{1,w} \leq C$ with $C > 0$. Then

$$\sup_{\lambda} \frac{1}{\lambda} \tau(e^{-\lambda^{-1}|T|^{-1}}) = \sup_{\lambda} \frac{1}{\lambda} \int_0^\infty e^{-\lambda^{-1}\mu_t(T)^{-1}} dt \leq \sup_{\lambda} \frac{1}{\lambda} \int_0^\infty e^{-\lambda^{-1}tC^{-1}} dt = C.$$

For the converse, we may assume that \mathcal{N} is either a type I factor with the standard trace or else that the trace τ is non-atomic. The general case will then follow by considering the embedding of \mathcal{N} into $\mathcal{N} \otimes L^\infty([0, 1])$.

So, suppose that $\sup_{\lambda} f_T(\lambda) = C$. By [10], in the case when \mathcal{N} is non-atomic, there exists \tilde{T} in \mathcal{N} such that $\mu_t(\tilde{T}) = \mu_t(T)\chi_{[0,t]}$. Define then $\tilde{f}_{T,t}(\lambda) := \frac{1}{\lambda} \tau(e^{-\lambda^{-1}\tilde{T}^{-1}})$. Then for all $t \geq 0$ we have

$$\sup_{\lambda} \frac{1}{\lambda} t e^{-\lambda^{-1}\mu_t(T)^{-1}} = \sup_{\lambda} \tilde{f}_{T,t}(\lambda) \leq \sup_{\lambda} f_T(\lambda) = C.$$

In particular, choosing $\lambda = \mu_t(T)^{-1}$ we find that $\mu_t(T) \leq Ce/t$. The case where \mathcal{N} is a type I factor is entirely similar. \square

Thus we have the curious situation that if h is unbounded, either one application of the Cesaro mean M , will produce a bounded function, or if not, then successive applications of the Cesaro mean will never produce a bounded function. Moreover, the boundedness of f_T singles out the ideal that arises in practise, namely $\mathcal{L}_{1,w}$. The use of Cesaro invariant functionals to produce Dixmier traces has enlarged the attention of noncommutative integration theory to \mathcal{Z}_1 , despite the fact that \mathcal{Z}_1 is unnatural from the point of view of most applications.

The distinction between boundedness and unboundedness of the function f_T is distinct from the issue of measurability, namely whether the value of a given Dixmier trace is independent

of the choice of (Cesaro invariant) functional ω . A sufficient condition for measurability is the existence of the limit $\lim_{\lambda \rightarrow \infty} (Mf_T)(\lambda)$, however whether this condition is necessary is not known except in special cases; see [24].

4. BANACH ALGEBRAS FOR NONUNITAL INTEGRATION

Much of the recent motivation for noncommutative integration theories comes from spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \tau)$ and their use in index theory. In Section 7, we review the definitions, but here just remind the reader that the commonest situations are when using the self-adjoint unbounded operator \mathcal{D} , the trace τ and suitable $s, p, t \in \mathbb{R}$, one (or more) of the maps

$$a \mapsto \tau(a(1 + \mathcal{D}^2)^{-s/2}), \quad a \mapsto \tau_\omega(a(1 + \mathcal{D}^2)^{-p/2}), \quad a \mapsto \tau(ae^{-t\mathcal{D}^2})$$

provides a sensible functional on the algebra \mathcal{A} . As we will explain, there are close links between these various situations.

Observe that for $p \geq 1$, the operator $(1 + \mathcal{D}^2)^{-p/2}$ is positive, injective and has norm ≤ 1 . These are the main properties we use and in the following text consider an operator G with these properties. In Section 5 we will also need some conditions on commutators $[G, a]$ between G and algebra elements a . In Section 7 we will verify these conditions for suitable spectral triples and $G = (1 + \mathcal{D}^2)^{-p/2}$.

4.1. Preliminaries. Let \mathcal{N} be a semifinite von Neumann algebra with τ , a faithful, normal, semifinite trace. Suppose also that G is a positive and injective operator in \mathcal{N} . We make neither τ -compactness nor summability hypotheses on G itself. Instead we consider that a natural condition of integrability for $a \in \mathcal{N}$, relative to G , would be to ask that

$$(4.1) \quad aG^s \in \mathcal{L}^1, \quad \forall s > 1.$$

If we consider $a \geq 0$ then the condition (4.1) is equivalent to

$$(4.2) \quad G^s a \in \mathcal{L}^1, \quad \forall s > 1,$$

since \mathcal{L}^1 is a $*$ -ideal. For $a \geq 0$ satisfying these equivalent conditions, it is shown by Bikchentaev in [3], that a also satisfies the equivalent conditions

$$(4.3) \quad G^{s/2} a G^{s/2} \in \mathcal{L}^1, \quad \Longleftrightarrow \quad a^{1/2} G^s a^{1/2} \in \mathcal{L}^1.$$

That is $(4.1) \Leftrightarrow (4.2) \Rightarrow (4.3)$, i.e. integrability of $a \geq 0$ implies square integrability of $a^{1/2}$. The symmetry of the conditions in (4.3) make them technically much easier to work with, however we point out the following.

Lemma 4.1. *The two equivalent conditions in (4.3) do not imply the two equivalent conditions (4.1) and (4.2).*

Proof. We provide a counterexample. Indeed, choose $\mathcal{D} \geq 1$ affiliated with \mathcal{N} and pick $T \in \mathcal{N}$, a projection, such that $T\mathcal{D}^{-1}T \in \mathcal{L}^1$ but with $\sqrt{T\mathcal{D}^{-1}T} \notin \mathcal{L}^1$. Now, define $\tilde{\mathcal{D}}$ affiliated with $M_2(\mathcal{N})$ and $\tilde{T} \in M_2(\mathcal{N})$, both acting on $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, by

$$\tilde{\mathcal{D}} := \begin{pmatrix} 1 & \sqrt{\mathcal{D}-1} \\ \sqrt{\mathcal{D}-1} & -1 \end{pmatrix}, \quad \tilde{T} := \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

Of course, $\tilde{\mathcal{D}}$ is invertible with

$$\tilde{\mathcal{D}}^{-1} = \begin{pmatrix} \mathcal{D}^{-1} & \sqrt{\mathcal{D}^{-1} - \mathcal{D}^{-2}} \\ \sqrt{\mathcal{D}^{-1} - \mathcal{D}^{-2}} & -\mathcal{D}^{-1} \end{pmatrix}.$$

We see at once that

$$\tilde{T} \tilde{\mathcal{D}}^{-1} \tilde{T} = \begin{pmatrix} T\mathcal{D}^{-1}T & 0 \\ 0 & 0 \end{pmatrix} = |\tilde{\mathcal{D}}^{-1} \tilde{T}|^2,$$

so that only $\tilde{T} \tilde{\mathcal{D}}^{-1} \tilde{T}$ belongs to $\mathcal{L}^1(M_2(\mathcal{N}), \tau \otimes \text{tr})$, not $\tilde{\mathcal{D}}^{-1} \tilde{T}$ (nor $\tilde{T} \tilde{\mathcal{D}}^{-1}$). \square

Earlier examples of this sort and additional information can be found in [4] and references therein. Replacing \mathcal{L}^1 with \mathcal{Z}_1 we obtain a counterexample for the corresponding statement in \mathcal{Z}_1 . Lemma 4.1 shows that when we formulate or obtain conditions in either of the symmetric forms in (4.3), *we can not get back to the unsymmetric forms in (4.1) and (4.2)*. By a result of one of us, [22, Lemma 5.10], similar results hold for a wide range of operator ideals, in particular for $a, G \in \mathcal{N}$ both positive we have

$$aG \in \mathcal{Z}_1 \Leftrightarrow Ga \in \mathcal{Z}_1 \not\Rightarrow a^{1/2}Ga^{1/2} \in \mathcal{Z}_1 \Leftrightarrow G^{1/2}aG^{1/2} \in \mathcal{Z}_1.$$

So again, whilst it is technically easier to work with the symmetrised conditions, we can not recover the unsymmetrised conditions from them. When we consider functionals on \mathcal{N} of the form

$$\mathcal{N}_+ \ni a \mapsto \tau(G^{s/2}aG^{s/2}), \quad s > 1,$$

we must confront the fact that we are dealing with a weight. When we determine the domain, it will not be an ideal, but rather a Hilbert algebra. We will not, however, go through the details of proving that our algebras satisfy the Hilbert algebra axioms as we do not need them here and will only work with a Banach algebra completion. Nevertheless we make use of some Hilbert algebra ideas and this explains in part the ‘square summable’ flavor of some of our hypotheses and results.

We also observe that the symmetric assumption ‘ $a^*G^sa \in \mathcal{L}^1$ for all $s > 1$ ’, equivalent to ‘ $G^sa \in \mathcal{L}^2$ for all $s > 1/2$ ’, is far easier to verify than the unsymmetrised condition ‘ $G^sa \in \mathcal{L}^1$ for all $s > 1$ ’. Indeed, an \mathcal{L}^2 -norm is typically easier to compute than an \mathcal{L}^1 -norm.

Moreover, in the type I case, that is for $\mathcal{N} = \mathcal{B}(\mathcal{H})$ equipped with the operator trace, there is a useful way to check if the product G^sa is Hilbert-Schmidt. This is a simple corollary of the ‘little Grothendieck Theorem’, the Grothendieck factorization principle (see [13]). The latter states that an operator is Hilbert-Schmidt if and only if it factors through a Banach space of the form

$L^1(X, \mu)$, for (X, μ) a σ -finite measure space, or of the form $C_0(K)$, for K a locally compact topological space. In particular, this applies for a large class of locally compact commutative spaces. Let M be a p -dimensional complete Riemannian manifold and Δ the associated scalar Laplace-Beltrami operator. Then, under mild assumptions of the behavior of the geometry at infinity (for instance those given in [19]), one readily deduces that the operator $M_f(1+\Delta)^{-ps/4}$, is Hilbert-Schmidt whenever $f \in L^2$ and $s > 1$. Indeed, M_f is obviously continuous from $L^\infty(M)$ (which is of the form $C_0(K)$ by the Gelfand-Naimark duality) to $L^2(M)$, while $(1+\Delta)^{-ps/4}$ continuously maps $L^2(M)$ to $L^\infty(M)$. This can be shown by standard heat-kernel methods (see [19, Lemma 4.5]).

4.2. Banach Algebras from zeta-function and heat-kernel.

Definition 4.2. *Given G , a positive and injective element of \mathcal{N} , define the two families of (possibly infinite) bilinear functionals on \mathcal{N}*

$$\zeta(a, b; s) := \tau(aG^s b), \quad g(a, b; \lambda) := \frac{1}{\lambda} \tau(ae^{-\lambda^{-1}G^{-1}} b), \quad a, b \in \mathcal{N}.$$

Then, introduce the following semi-norms on \mathcal{N}

$$\|a\|_\zeta := \sup_{1 \leq s \leq 2} \sqrt{s-1} \zeta(a^*, a; s)^{1/2}, \quad \|a\|_{\text{HK}} := \|Mg(a^*, a; \cdot)\|_\infty^{1/2},$$

where M denotes the Césaro mean of the multiplicative group \mathbb{R}_+^ .*

Note that since $\langle a, b \rangle_{\zeta, s} := \zeta(a^*, b; s)$ and $\langle a, b \rangle_{\text{HK}, \lambda} := g(a^*, b; \lambda)$ are inner products, it follows at once that $\|\cdot\|_\zeta$ and $\|\cdot\|_{\text{HK}}$ are positively homogeneous and satisfy the triangle inequality. But they are also injective maps. Indeed, if for instance, $\|a\|_\zeta = 0$, then by the faithfulness of τ , we deduce that $0 = a^*G^s a = |G^{s/2}a|^2$ and thus $G^{s/2}a = 0$ too. Then, from the injectivity of G , we get that $a = 0$. A similar result holds for $\|a\|_{\text{HK}}$. This shows that $\|\cdot\|_\zeta$ and $\|\cdot\|_{\text{HK}}$ are true norms, not only semi-norms.

By the functional calculus, we see that if $a^*G^s a$ is trace class for some $s > 1$, so is $a^*e^{-tG^{-1}}a$ for all $t > 0$. Indeed, we have the operator inequality

$$(4.4) \quad a^*e^{-tG^{-1}}a \leq \|G^{-s}e^{-tG^{-1}}\| a^*G^s a \leq (s/t)^s e^{-s} a^*G^s a.$$

The finiteness of such semi-norms is closely related to the zeta function and heat kernel characterizations of the ideal \mathcal{Z}_p in the unital case, [8].

Proposition 4.3. *Let $B_\zeta(G)$ (respectively $B_{\text{HK}}(G)$) be the normed subset of \mathcal{N} relative to the norm $\|a\|_\zeta + \|a^*\|_\zeta + \|a\|$ (respectively $\|a\|_{\text{HK}} + \|a^*\|_{\text{HK}} + \|a\|$). Then, $B_\zeta(G)$ and $B_{\text{HK}}(G)$ are Banach $*$ -algebras.*

When no confusion can occur, we write B_ζ and B_{HK} instead of $B_\zeta(G)$ and $B_{\text{HK}}(G)$.

Proof of Proposition 4.3. It is clear that B_ζ and B_{HK} are normed linear spaces with symmetric norms. Let us first show that they are sub-multiplicative. For a positive numerical function f , we have:

$$\tau((ab)^* f(G) ab) \leq \|b\|^2 \|a^* f(G) a\|_1 = \|b\|^2 \tau(a^* f(G) a).$$

This clearly entails that $\|ab\|_\zeta \leq \|b\| \|a\|_\zeta$ and $\|(ab)^*\|_\zeta \leq \|a\| \|b^*\|_\zeta$, and thus

$$\begin{aligned} \|ab\|_\zeta + \|(ab)^*\|_\zeta + \|ab\| &\leq \|b\| \|a\|_\zeta + \|a\| \|b^*\|_\zeta + \|a\| \|b\| \\ &\leq (\|a\|_\zeta + \|a^*\|_\zeta + \|a\|) (\|b\|_\zeta + \|b^*\|_\zeta + \|b\|). \end{aligned}$$

A similar statement holds for the norm of B_{HK} .

It remains to prove that B_ζ and B_{HK} are complete for the norm topology we have introduced. But this is an easy exercise since a Cauchy sequence $(a_n)_{n \in \mathbb{N}}$ in B_ζ is convergent as an element of \mathcal{N} (since the norm of B_ζ dominates the one of \mathcal{N}). Call $a \in \mathcal{N}$ such a limit and suppose that $\|a\|_\zeta = \infty$, i.e. $\lim \|a_n\|_\zeta = \infty$. This is a contradiction with the assumption of $(a_n)_{n \in \mathbb{N}}$ being B_ζ -Cauchy since by the second triangle inequality we have $|\|a_n\|_\zeta - \|a_m\|_\zeta| \leq \|a_n - a_m\|_\zeta$. The arguments for B_{HK} are similar. \square

The algebras B_ζ , B_{HK} need not be uniformly closed, nor weakly closed, nor be ideals (even one-sided) in \mathcal{N} . We now prove that these two notions of ‘square integrability’ in fact coincide.

Theorem 4.4. *The norms $\|\cdot\|_\zeta$ and $\|\cdot\|_{\text{HK}}$ are equivalent.*

Proof. Fix any $a \in \mathcal{N}$ with $\|a\|_\zeta < \infty$. We need to show that the associated function $g(a^*, a; \cdot)$ has bounded Cesaro mean.

$$\begin{aligned} (Mg(a^*, a; \cdot))(x) &= \frac{1}{\log x} \int_1^x g(a^*, a; \lambda) \frac{d\lambda}{\lambda} = \frac{1}{\log x} \int_1^x \frac{1}{\lambda} \tau(a^* e^{-\lambda^{-1} G^{-1}} a) \frac{d\lambda}{\lambda} \\ &= \frac{1}{\log x} \int_{x^{-1}}^1 \tau(a^* e^{-t G^{-1}} a) dt = \frac{1}{\log x} \tau(a^* G e^{-x^{-1} G^{-1}} a) - \frac{1}{\log x} \tau(a^* G e^{-G^{-1}} a) \\ &\leq \frac{1}{\log x} \tau(a^* G e^{-x^{-1} G^{-1}} a). \end{aligned}$$

The inversion of the trace and the integral can be justified by Fubini’s Theorem. Indeed, the operator-valued function $t \mapsto \chi_{[x^{-1}, 1]}(t) a^* e^{-t G^{-1}} a$, belongs to $C([x^{-1}, 1], \mathcal{L}^1(\mathcal{N}, \tau))$ and thus to $L^1 \cap L^\infty([x^{-1}, 1], \mathcal{L}^1(\mathcal{N}, \tau))$. Using the embedding of $C([x^{-1}, 1], \mathcal{N})$ into the von Neumann algebra $L^\infty([x^{-1}, 1], dt) \otimes \mathcal{N}$, we see that it defines an integrable element of $L^\infty([0, 1], dt) \otimes \mathcal{N}$ for the faithful semifinite trace $\int \otimes \tau$. Now, making the change of variable $x = e^{1/\varepsilon}$ (i.e. $\varepsilon = s - 1$) in the previous expression, we obtain, for $0 < \varepsilon < 1$,

$$|(Mg(a^*, a; \cdot))(e^{1/\varepsilon})| \leq \varepsilon \tau(a^* G e^{-e^{-1/\varepsilon} G^{-1}} a) \leq \varepsilon \tau(a^* G^{1+\varepsilon} a) \|G^{-\varepsilon} e^{-e^{-1/\varepsilon} G^{-1}}\|,$$

where we have used the operator inequality

$$a^* G e^{-e^{-1/\varepsilon} G^{-1}} a \leq \|G^{-\varepsilon} e^{-e^{-1/\varepsilon} G^{-1}}\| a^* G^{1+\varepsilon} a.$$

But the function $x \mapsto x^\varepsilon e^{-e^{-1/\varepsilon}x}$ reaches its maximum at $x = \varepsilon e^{1/\varepsilon}$, where its value is $e\varepsilon^\varepsilon e^{-\varepsilon} \leq e$. Thus,

$$|(Mg(a^*, a; \cdot))(e^{1/\varepsilon})| \leq e\varepsilon \tau(a^* G^{1+\varepsilon} a) = e\varepsilon \zeta(a^*, a; 1 + \varepsilon),$$

which concludes the proof of the first inclusion, since the right hand side is bounded by assumption.

To show that $\|\cdot\|_\zeta \leq C\|\cdot\|_{\text{HK}}$, we will use the Laplace transform. Fix any $a \in \mathcal{N}$ with finite HK-norm. We have, writing again $\varepsilon = s - 1$,

$$G^{1+\varepsilon} = \frac{1}{\Gamma(1+\varepsilon)} \int_0^\infty t^\varepsilon e^{-tG^{-1}} dt.$$

Disregarding the bounded pre-factor $\Gamma(1+\varepsilon)^{-1}$ in the expression above, we first decompose the integral into two pieces: $\int_0^\infty = \int_0^1 + \int_1^\infty$, to write $a^* G^{1+\varepsilon} a$ as a sum of two operators. For the second term, we obtain

$$\begin{aligned} \tau\left(\int_1^\infty t^\varepsilon a^* e^{-tG^{-1}} a dt\right) &\leq \int_1^\infty t^\varepsilon \tau(a^* e^{-G^{-1}} a) \|e^{-(t-1)G^{-1}}\| dt \\ &= g(a^*, a; 1) \int_1^\infty t^\varepsilon e^{-(t-1)\|G\|^{-1}} dt \leq C_1, \end{aligned}$$

where the constant C_1 is independent of ε . For the first term, we can exchange the trace and the integral because of the finite range of the integration. This reads,

$$\tau\left(\int_0^1 t^\varepsilon a^* e^{-tG^{-1}} a dt\right) = \int_0^1 t^\varepsilon \tau(a^* e^{-tG} a) dt = \int_1^\infty \lambda^{-\varepsilon} g(a^*, a; \lambda) \frac{d\lambda}{\lambda}.$$

Now, let us again decompose the integral as $\int_1^\infty = \int_1^{e^{\varepsilon-1}} + \int_{e^{\varepsilon-1}}^\infty$. For the first part, we can conclude that

$$\int_1^{e^{\varepsilon-1}} \lambda^{-\varepsilon} g(a^*, a; \lambda) \frac{d\lambda}{\lambda} \leq \int_1^{e^{\varepsilon-1}} g(a^*, a; \lambda) \frac{d\lambda}{\lambda} = \frac{1}{\varepsilon} (Mg(a^*, a; \cdot))(e^{\varepsilon-1}) \leq \frac{\|Mg(a^*, a; \cdot)\|_\infty}{\varepsilon} =: \frac{C_2}{\varepsilon}.$$

The last term is

$$(4.5) \quad \int_{e^{\varepsilon-1}}^\infty \lambda^{-\varepsilon} g(a^*, a; \lambda) \frac{d\lambda}{\lambda}.$$

Now, we are going to make use of the following change of variable: $0 \leq y(\lambda) := \int_1^\lambda g(a^*, a; \sigma) \frac{d\sigma}{\sigma}$, a monotonically increasing function of λ . Observing that $y(\lambda) = (Mg(a^*, a; \cdot))(\lambda) \log(\lambda)$, there exists a positive constant C_3 , such that $y(\lambda) \leq C_3 \log(\lambda)$, and thus $\lambda^{-\varepsilon} \leq e^{-\varepsilon C_3^{-1} y}$. This implies that the expression (4.5) is smaller than

$$\int_0^\infty e^{-\varepsilon C_3^{-1} y} dy = \varepsilon^{-1} \int_0^\infty e^{-C_3^{-1} y} dy =: \frac{C_3}{\varepsilon}.$$

Gathering these estimates together proves that

$$\varepsilon \tau(a^* G^{1+\varepsilon} a) \leq \varepsilon \left(C_1 + \frac{C_2}{\varepsilon} + \frac{C_3}{\varepsilon} \right),$$

and thus the set $\{(s-1)\zeta(a^*, a; s) : 1 \leq s \leq 2\}$ is bounded and so $\|a\|_\zeta < \infty$. This concludes the proof of the proposition. \square

Applying this result to the unital case, i.e. when G alone belongs to \mathcal{Z}_1 , and combining it with Theorem 2.1, we get an interesting fact.

Corollary 4.5. *The following three norms on \mathcal{Z}_1 are equivalent.*

$$\|T\|_{1,\infty} = \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds, \quad \sup_{s>1} (s-1) \tau(|T|^s), \quad \sup_{\lambda>0} \frac{1}{\log(\lambda)} \int_1^\lambda \mu^{-2} \tau(e^{-\mu^{-1}|T|^{-1}}) d\mu.$$

A positive element $\omega \in L^\infty(\mathbb{R})_+^*$, is called a Banach limit, if it is translation invariant and satisfies $\text{ess} - \liminf_{t \rightarrow \infty} (f) \leq \omega(f) \leq \text{ess} - \limsup_{t \rightarrow \infty} (f)$, for all $0 \leq f \in L^\infty(\mathbb{R})$. In particular, we see that any Banach limit must vanish on $C_0(\mathbb{R})$. Given such a Banach limit ω , we define the dilation invariant functional $\tilde{\omega}$ on \mathbb{R}_+^* , by $\tilde{\omega} := \omega \circ \log$. We use the notation $\omega - \lim_{t \rightarrow \infty} f(t)$ instead of $\omega(f)$. Next we show that the $\tilde{\omega}$ -residue of the zeta function $\zeta(a^*, a; \cdot)$ coincides with the ω -limit of the Césaro mean of the heat-trace function $g(a^*, a; \cdot)$.

Theorem 4.6. *Let $a \in B_\zeta$, then for any Banach limit ω , we have*

$$\omega - \lim_{\lambda \rightarrow \infty} M(g(a^*, a; \cdot))(\lambda) = \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \zeta(a^*, a; 1 + \frac{1}{r}).$$

If moreover ω is M -invariant and $g(a, a; \cdot)$ is bounded, then

$$\omega - \lim_{\lambda \rightarrow \infty} g(a^*, a; \lambda) = \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \zeta(a^*, a; 1 + \frac{1}{r}).$$

In both these situations, if one of the ordinary limits exists, both limits exist, and they coincide.

Proof. We only prove the first part, the remaining statements will then follow immediately from general properties of Banach limits.

In the course of the proof of Theorem 4.4, we have shown that

$$\left\| a^* G^{1+\frac{1}{r}} a - \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^1 t^{\frac{1}{r}} a^* e^{-tG^{-1}} a dt \right\|_1 \leq g(a^*, a; 1) \int_1^\infty t^{\frac{1}{r}} e^{-(t-1)\|G\|^{-1}} dt.$$

Hence

$$\lim_{r \rightarrow \infty} \frac{1}{r} \left[\tau(a^* G^{1+\frac{1}{r}} a) - \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^1 t^{\frac{1}{r}} \tau(a^* e^{-tG^{-1}} a) dt \right] = 0,$$

thus setting $\tilde{\omega} = \omega \circ \log$, we have:

$$\tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \tau(a^* G^{1+\frac{1}{r}} a) = \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^1 t^{\frac{1}{r}} \tau(a^* e^{-tG^{-1}} a) dt.$$

Substituting $t = e^{-\mu}$, a little computation shows that

$$(4.6) \quad \int_0^1 t^{\frac{1}{r}} \tau(a^* e^{-tG^{-1}} a) dt = \int_0^\infty e^{-\frac{\mu}{r}} d\beta(\mu) \quad \text{where} \quad \beta(\mu) = \int_0^\mu e^{-\nu} \tau(a^* e^{-e^{-\nu} G^{-1}} a) d\nu.$$

We are now wish to use the weak-* Karamata Theorem [7, Theorem 2.2], and need to check the various hypotheses. First, ω is a translation invariant mean on \mathbb{R} , so $\tilde{\omega}$ is a dilation invariant mean on \mathbb{R}_+^* . Next, β is positive, increasing and continuous on \mathbb{R}^+ , and satisfies $\beta(0) = 0$. Finally, we need to check that $\int_0^\infty e^{-\frac{\mu}{r}} d\beta(\mu)$ is finite for any $r > 0$. But this follows immediately from the first equality in Equation (4.6). Hence, the weak-* Karamata Theorem gives us

$$\tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^\infty e^{-\frac{\mu}{r}} d\beta(\mu) = \tilde{\omega} - \lim_{\mu \rightarrow \infty} \frac{\beta(\mu)}{\mu}.$$

But

$$\frac{\beta(\mu)}{\mu} = \frac{1}{\mu} \int_1^{e^\mu} \lambda^{-2} \tau(a^* e^{-\lambda^{-1} G^{-1}} a) d\lambda = (Mg(a^*, a; \cdot))(e^\mu),$$

from which the result follows, since

$$\tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \tau(a G^{1+\frac{1}{r}} a) = \tilde{\omega} - \lim_{\mu \rightarrow \infty} (Mg(a, a; \cdot)) \circ \exp(\mu) = \omega - \lim_{\mu \rightarrow \infty} (Mg(a, a; \mu)). \quad \square$$

To conclude this discussion, we give some obvious but useful stability properties of B_ζ .

Lemma 4.7. *i) If $a \in B_\zeta$ and $f \in L^\infty(\mathbb{R})$, then $af(G) \in B_\zeta$, and when $a^* = a$, $af(a) \in B_\zeta$.
ii) If $0 \leq a, b \in \mathcal{N}$ are such that $a \in B_\zeta$ and $b^2 \leq a^2$, then $b \in B_\zeta$.
iii) If $a \in B_\zeta$, then $|a|, |a^*| \in B_\zeta$.*

Proof. From the ideal property of the trace-norm we obtain

$$\tau(\bar{f}(G) a^* G^s a f(G)) \leq \|f\|_\infty^2 \tau(a^* G^s a),$$

and from the operator inequality

$$af(G) G^s \bar{f}(G) a^* \leq \|f\|_\infty^2 a G^s a^*,$$

we obtain the first part of *i)*. The second part of *i)* is even more immediate.

To prove *ii)*, note that from (4.3), that is from the trace property, we have

$$\tau(b G^s b) = \tau(G^{s/2} b^2 G^{s/2}) \leq \tau(G^{s/2} a^2 G^{s/2}) = \tau(a G^s a).$$

Finally, to obtain *iii)*, let $u|a|$ and $v|a^*|$ be the polar decomposition of a and a^* . Then of course, $|a| = u^* a = a^* u$ and $|a^*| = v^* a^* = av$. Thus,

$$\tau(|a| G^s |a|) = \tau(u^* a G^s a^* u) \leq \tau(a G^s a^*).$$

The proof for $|a^*|$ is entirely similar. \square

4.3. Relations between $B_\zeta(G)$, \mathcal{Z}_p and $\mathcal{L}^{p,\infty}$.

Proposition 4.8. *If $a, b \in B_\zeta(G)$, then $bGa \in \mathcal{Z}_1$. Moreover, when $b = a^*$, we have the following partial-trace estimate:*

$$\sigma_t(a^*Ga) := \int_0^t \mu_s(a^*Ga) ds \leq \frac{1}{e} \|a^* e^{-G^{-1}} a\|_1 + \|a\|^2 + \|Mg(a^*, a; \cdot)\|_\infty \log(1+t).$$

Proof. By [10, Lemma 2.3] we have the inequality:

$$\|bGa\|_{1,\infty} \leq \| |bG^{1/2}|^2 \|_{1,\infty}^{1/2} \| |G^{1/2}a|^2 \|_{1,\infty}^{1/2} = \| |G^{1/2}b^*|^2 \|_{1,\infty}^{1/2} \| |G^{1/2}a|^2 \|_{1,\infty}^{1/2} = \|bGb^*\|_{1,\infty}^{1/2} \|a^*Ga\|_{1,\infty}^{1/2},$$

so we may assume without loss of generality that $b = a^*$.

Recall from [7, Lemma 3.3] that if $a \in \mathcal{N}$ has norm $\|a\| \leq M$, then for any $1 \leq s < 2$,

$$(a^*Ga)^s \leq M^{2(s-1)} a^*G^s a.$$

Using this inequality we have

$$\limsup_{s \rightarrow 1} (s-1) \tau((a^*Ga)^s) \leq \limsup_{s \rightarrow 1} \|a\|^{2(1-s)} (s-1) \tau(a^*G^s a) < \infty,$$

which shows that $a \in B_\zeta \Rightarrow a^*Ga \in \mathcal{Z}_1$.

To obtain the estimate of the partial trace, we use the Laplace transform to write

$$a^*Ga = \int_0^\infty a^* e^{-tG^{-1}} a dt = \left(\int_0^{e^{-k}} + \int_{e^{-k}}^1 + \int_1^\infty \right) a^* e^{-tG^{-1}} a dt =: C_k + B_k + A.$$

By (4.4), we see that A is trace-class with $\|A\|_1 \leq e^{-1} \|a^* e^{-G^{-1}} a\|_1$ and we focus on the rest. For C_k , we have the bound $\|C_k\| \leq \|a\|^2 e^{-k}$. The operator B_k can be bounded in trace-norm:

$$\begin{aligned} \|B_k\|_1 &\leq \int_{e^{-k}}^1 \tau(a^* e^{-tG^{-1}} a) dt = \int_1^{e^k} g(a^*, a; \lambda) \frac{d\lambda}{\lambda} = \ln(e^k) (Mg(a^*, a; \cdot))(e^k) \\ &\leq \|Mg(a^*, a; \cdot)\|_\infty k. \end{aligned}$$

Recall the K -functional associated to the Banach couple $(\mathcal{L}^1, \mathcal{N})$

$$K(T, t; \mathcal{L}^1, \mathcal{N}) := \inf \{ \|T_1\|_1 + t\|T_2\|, T = T_1 + T_2, T_1 \in \mathcal{L}^1, T_2 \in \mathcal{N} \}.$$

It is known that in this case, it can be exactly evaluated and reads

$$K(T, t; \mathcal{L}^1, \mathcal{N}) = \int_0^t \mu_s(T) ds =: \sigma_t(T).$$

Thus, writing $D = C_k + B_k$, we can estimate

$$\sigma_s(D) \leq \|B_k\|_1 + s\|C_k\| \leq \|Mg(a^*, a; \cdot)\|_\infty k + \|a\|^2 s e^{-k}.$$

Finally, given $s \in (0, \infty)$, define $k \in \mathbb{R}_+$ as $k = \ln(1+s)$, so that we have

$$\sigma_s(D) \leq \|Mg(a^*, a; \cdot)\|_\infty \ln(1+s) + \|a\|^2 s(1+s)^{-1}.$$

Gathering these estimates together, we find the bound stated in the lemma. \square

The next result refines our approach to obtain containments in \mathcal{Z}_q , $q \geq 1$.

Proposition 4.9. *Let $\delta \in (0, 1]$ and $0 \leq a \in \mathcal{N}$ be such that $aGa \in \mathcal{Z}_1$. Then, for any $\varepsilon \in (0, \delta/2]$, $a^\delta G^\varepsilon \in \mathcal{Z}_{1/\varepsilon}$ with*

$$\|a^\delta G^\varepsilon\|_{1/\varepsilon, \infty} \leq \|a\|^{\delta-2\varepsilon} \|aGa\|_{1, \infty}^\varepsilon.$$

From Theorem 4.8, we see that the assumption $aGa \in \mathcal{Z}_1$ is satisfied for $a \in B_\zeta$.

Proof of Proposition 4.9. Note that the statement is equivalent to:

$$aGa \in \mathcal{Z}_1 \Rightarrow a^\delta G^\varepsilon a^\delta \in \mathcal{Z}_{1/\varepsilon}, \quad \forall \varepsilon \in (0, \delta].$$

Consider the holomorphic operator valued function on the open strip $S = \{z \in \mathbb{C} : \Re z \in (0, 1)\}$, given by $F(z) = a^z G^z a^z$. For all $y \in \mathbb{R}$, we have $F(iy) \in \mathcal{N}$ with $\|F(iy)\| \leq 1$. Moreover, $F(1 + iy) \in \mathcal{Z}_1$. Indeed, since

$$\int_0^t \mu_s(AB) ds \leq \int_0^t \mu_s(A) \mu_s(B) ds, \quad \forall A, B \in \mathcal{N},$$

proven in [17, Theorem 4.2, iii)] and [10, Proposition 1.1], we obtain

$$\begin{aligned} \int_0^t \mu_s(F(1 + iy)) ds &= \int_0^t \mu_s(a^{1+iy} G^{1+iy} a^{1+iy}) ds \leq \int_0^t \mu_s(a G^{1+iy} a) ds \\ &\leq \int_0^t \mu_s(a G^{1/2}) \mu_s(G^{1/2+iy} a) ds \leq \int_0^t \mu_s(a G^{1/2}) \mu_s(G^{1/2} a) ds = \int_0^t \mu_s(aGa) ds, \end{aligned}$$

and thus $\|F(1 + iy)\|_{1, \infty} \leq \|aGa\|_{1, \infty}$.

This shows that $a^\varepsilon G^\varepsilon a^\varepsilon$ belongs to the first complex interpolation space $(\mathcal{Z}_1, \mathcal{N})_{[\varepsilon]}$ and hence belongs to $(\mathcal{Z}_1, \mathcal{N})^{[\varepsilon]}$, the second complex interpolation space. (There is a norm one injection from $(X_0, X_1)_{[\theta]}$ into $(X_0, X_1)^{[\theta]}$, for any Banach couple (X_0, X_1) , $\theta \in [0, 1]$). But the latter is $\mathcal{Z}_{1/\varepsilon}$, as shown in subsection 2.3. In summary, we have

$$\|a^\varepsilon G^\varepsilon a^\varepsilon\|_{1/\varepsilon, \infty} = \|F(\varepsilon)\|_{(\mathcal{Z}_1, \mathcal{N})^{[1-\varepsilon]}} \leq \|F(\varepsilon)\|_{(\mathcal{Z}_1, \mathcal{N})_{[1-\varepsilon]}} \leq \|F(0)\|^{1-\varepsilon} \|F(1)\|^\varepsilon_{1, \infty} = \|aGa\|_{1, \infty}^\varepsilon,$$

and from the ideal property, we obtain the announced result

$$\|a^\delta G^\varepsilon a^\delta\|_{1/\varepsilon, \infty} \leq \|a\|^{2(\delta-\varepsilon)} \|aGa\|_{1, \infty}^\varepsilon, \quad \forall \delta \in (0, 1], \quad \forall \varepsilon \in (0, \delta]. \quad \square$$

According to Lemma 4.1, the assumption that $a, b \in B_\zeta$ is not enough to ensure that abG belongs to \mathcal{Z}_1 . On a more positive note, the intuitive result that when $g(a^*, a; \cdot)$ is already bounded, that $a^* G^\varepsilon a$, $\varepsilon \in (0, 1)$, is in the small ideal $\mathcal{L}^{1/\varepsilon, \infty}$ is true.

Proposition 4.10. *Let $\varepsilon \in (0, 1)$ and let $a \in \mathcal{N}$ be such that the map $\mathbb{R}^+ \ni \lambda \mapsto g(a^*, a; \lambda)$ is bounded. Then $a^* G^\varepsilon a \in \mathcal{L}^{1/\varepsilon, \infty}$, with*

$$(4.7) \quad \mu_s(a^* G^\varepsilon a) \leq \frac{1}{\Gamma(\varepsilon)} \left(\frac{1}{\varepsilon} \|a\|^2 + \frac{1}{1-\varepsilon} \|g(a^*, a; \cdot)\|_\infty \right) s^{-\varepsilon}.$$

Proof. We write

$$a^* G^\varepsilon a = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty t^{\varepsilon-1} a^* e^{tG^{-1}} a dt,$$

and split $\int_0^\infty = \int_{e^{-k}}^\infty + \int_0^{e^{-k}}$, to obtain $a^* G^\varepsilon a = B_k + C_k$, $k \in \mathbb{R}_+^*$. We notice that for any $S \in \mathcal{N}$,

$$\|S\|_1 = \int_0^\infty \mu_t(S) dt \geq \int_0^s \mu_t(S) dt \geq s \mu_s(S).$$

Then, using Fan's inequality, we obtain

$$\mu_s(B_k + C_k) \leq \mu_0(C_k) + \mu_s(B_k) \leq \|C_k\| + s^{-1} \|B_k\|_1.$$

We have first $\|C_k\| \leq C \|a\|^2 e^{-\varepsilon k}$. Indeed

$$\|C_k\| \leq \frac{1}{\Gamma(\varepsilon)} \|a\|^2 \int_0^{e^{-k}} t^{\varepsilon-1} dt = \frac{1}{\varepsilon \Gamma(\varepsilon)} \|a\|^2 e^{-\varepsilon k}.$$

For the second part, we have

$$\begin{aligned} \|B_k\|_1 &\leq \frac{1}{\Gamma(\varepsilon)} \int_{e^{-k}}^\infty \tau(a^* e^{-tG^{-1}} a) t^{\varepsilon-1} dt = \frac{1}{\Gamma(\varepsilon)} \int_{e^{-k}}^\infty g(a^*, a; t^{-1}) t^{\varepsilon-2} dt \\ &\leq \frac{1}{\Gamma(\varepsilon)} \|g(a^*, a; \cdot)\|_\infty (1 - \varepsilon)^{-1} e^{k(1-\varepsilon)}. \end{aligned}$$

Thus we have

$$\mu_s(B_k + C_k) \leq \frac{1}{\Gamma(\varepsilon)} \left(\frac{1}{\varepsilon} \|a\|^2 + \frac{1}{1-\varepsilon} \|g(T^*, T; \cdot)\|_\infty \frac{e^k}{s} \right) e^{-\varepsilon k}.$$

So, if for each $s \in \mathbb{R}^+$, we choose $k = \log s$, we obtain the desired estimate. \square

Remark. By polarization, the positivity assumption in Proposition 4.10 may be eliminated. That is, for $a, b \in B_\zeta$, both having bounded g -function, we have $aG^\varepsilon b \in \mathcal{L}^{1/\varepsilon, \infty}$, for all $\varepsilon \in (0, 1)$. Note finally the singular nature of the estimate in equation (4.7) in the limit $\varepsilon \rightarrow 1^-$. This prevents us reaching the weak- \mathcal{L}^1 -space, $\mathcal{L}_{1,w}$ of definition 3.1, with these techniques.

5. ZETA FUNCTIONS AND DIXMIER TRACES

This Section contains the main application of our previous results. We are interested in the question, first raised in [11, Chapter 4], and further studied in considerable detail in [7–9, 22–24, 35] in the unital case, concerning the relationship between singularities of the zeta function and the Dixmier trace. The extension of this result to the nonunital case without appealing to the existence of local units has interested a number of authors. The construction of our Banach algebras B_ζ was motivated by this question. The next Section collects some general lemmas needed later.

5.1. General facts. We first prove a result which allows us to manipulate the commutator of fractional powers. We are indebted to Alain Connes for communicating the proof to us, which we reproduce here for completeness.

Lemma 5.1. *Let $0 \leq A, B \in \mathcal{N}$ be such that $[A, B] \in \mathfrak{S}$, where \mathfrak{S} denotes any symmetrically normed (or quasi-normed) ideal of \mathcal{N} . Denoting by \mathfrak{S}_p , $p \geq 1$, the p -convexification of \mathfrak{S} , for all $\alpha, \beta \in (0, 1]$, we have $[A^\alpha, B^\beta] \in \mathfrak{S}_{1/\alpha\beta}$, with*

$$\|[A^\alpha, B^\beta]\|_{\mathfrak{S}_{1/\alpha\beta}} \leq \|A\|^{\alpha(1-\beta)} \|B\|^{\beta(1-\alpha)} \|[A, B]\|_{\mathfrak{S}}^{\alpha\beta}.$$

Proof. By homogeneity, we can assume that $\|A\| = \|B\| = 1$. We are going to use the Cayley transform twice to obtain a commutator estimate from a difference estimate and then use the BKS inequality [5]. To this end, let U be the unitary operator $U := (i + B)(i - B)^{-1}$. A quick computation shows that $[A, U] = 2i(i - B)^{-1}[A, B](i - B)^{-1}$, which gives

$$U^*AU - A = U^*[A, U] = 2iU^*(i - B)^{-1}[A, B](i - B)^{-1}.$$

Thus, we see that $\|U^*AU - A\|_{\mathfrak{S}_p} = \|[A, B]\|_{\mathfrak{S}_p}$. Using finally that $(U^*AU)^\alpha = U^*A^\alpha U$, $\forall \alpha > 0$, and the BKS inequality $\|X^\alpha - Y^\alpha\|_{\mathfrak{S}_{1/\alpha}} \leq \|X - Y\|_{\mathfrak{S}}^\alpha$, we obtain

$$[A, B] \in \mathfrak{S} \Leftrightarrow U^*AU - A \in \mathfrak{S} \Rightarrow U^*A^\alpha U - A^\alpha \in \mathfrak{S}_{1/\alpha} \Leftrightarrow [A^\alpha, B] \in \mathfrak{S}_{1/\alpha}.$$

One concludes the proof using the same trick with the unitary $V := (i + A^\alpha)(i - A^\alpha)^{-1}$. \square

Lemma 5.2. *Let $A, B \in \mathcal{N}$, $B^* = B$, such that $[A, B] \in \mathfrak{S}$ for any symmetrically normed ideal of \mathcal{N} and let $\varphi \in C_c^\infty(\mathbb{R})$. Then $[A, \varphi(B)] \in \mathfrak{S}$ with*

$$\|[A, \varphi(B)]\|_{\mathfrak{S}} \leq \|\widehat{\varphi}'\|_1 \|[A, B]\|_{\mathfrak{S}}.$$

Proof. Since φ is a smooth compactly supported function, it is the Fourier transform of a Schwartz function $\widehat{\varphi}$ and thus $\varphi(B) = \int_{\mathbb{R}} \widehat{\varphi}(\xi) e^{-2i\pi\xi B} d\xi$. The result then follows from the identity

$$[A, e^{-2i\pi\xi B}] = -2i\pi\xi \int_0^1 e^{-2i\pi\xi sB} [A, B] e^{-2i\pi\xi(1-s)B} ds. \quad \square$$

Stronger estimates than that given above are available from [30, 31]. Finally we will need

Lemma 5.3. *i) If $T \in \mathcal{Z}_1^0$, then $\lim_{\varepsilon \rightarrow 0^+} \varepsilon \|T\|_{1+\varepsilon} = 0$.*

ii) Let $a \in B_\zeta$ and $\delta \in (0, 1]$. Then the map $(0, \delta/2) \ni \varepsilon \mapsto \|a^\delta G^\varepsilon\|_{1+1/\varepsilon}$ is bounded.

Proof. The first claim follows from [8, Theorem 4.5 i)].

To prove the second part, note that for an arbitrary $T \in \mathcal{Z}_1$, by the definition of the norm in the Marcinkiewicz space \mathcal{Z}_1 , we have $\mu_t(T) \prec\prec \|T\|_{1,\infty}/(1+t)$. Since the Schatten spaces \mathcal{L}^p , $1 \leq p \leq \infty$, are fully symmetric operator spaces we thus have

$$\|T\|_p \leq \|T\|_{1,\infty} \|t \mapsto (1+t)^{-1}\|_p, \quad p > 1,$$

that is

$$(5.1) \quad \|T\|_p \leq \|T\|_{1,\infty} (p-1)^{-1/p}.$$

Let $T := (a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}$. This operator belongs to \mathcal{Z}_1 , because $aGa \in \mathcal{Z}_1$ by Theorem 4.8 and thus $a^\delta G^{2\varepsilon} a^\delta \in \mathcal{Z}_{1/2\varepsilon}$ by Proposition 4.9. Applying the estimate (5.1), with $p = 1 + \varepsilon$, to this operator yields

$$(5.2) \quad \begin{aligned} \|G^\varepsilon a^\delta\|_{1+1/\varepsilon} &= \|(a^\delta G^{2\varepsilon} a^\delta)^{1/2+1/2\varepsilon}\|_1^{\varepsilon/(1+\varepsilon)} = \|(a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}\|_{1+\varepsilon}^\varepsilon \\ &\leq \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|(a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}\|_{1,\infty}^\varepsilon = \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty}^{1/2}. \end{aligned}$$

But Proposition 4.9 gives also the inequality

$$(5.3) \quad \|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty} \leq \|a\|^{2(\delta-2\varepsilon)} \|aGa\|_{1,\infty}^{2\varepsilon}, \quad \forall \delta \in (0, 1], \quad \forall \varepsilon \in (0, \delta/2).$$

Combining (5.2) with (5.3), we get

$$\|G^\varepsilon a^\delta\|_{1+1/\varepsilon} \leq \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty}^{1/2} \leq \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|a\|^{\delta-2\varepsilon} \|aGa\|_{1,\infty}^\varepsilon.$$

This proves the claim since $\varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \rightarrow 1$. \square

5.2. Approximation schemes. Without loss of generality we may assume throughout that $G^{-1} \geq 1$. In the following, we fix $0 \leq a \in \mathcal{N}$ and we assume further that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. We stress that while purely technical at the first glance, this extra δ -condition turns out to be the key assumption to get an equivalence $a \in B_\zeta \Leftrightarrow a^*Ga \in \mathcal{Z}_1$. This is explained in Section 6. We then construct a pair of approximation processes for a in the strong topology, the first being given by

$$a_n := aP_n, \quad \text{where} \quad P_n := \int_{1/n}^{\|a\|} dE_a(\lambda),$$

with $\int_0^{\|a\|} \lambda dE_a(\lambda)$ the spectral resolution of a . Note the operator inequality $a_n^2 \geq \frac{1}{n^2} P_n$. Lemma 4.7 ii) implies then that $P_n \in B_\zeta$ as well.

Next, for each $n \in \mathbb{N}$, we pick $0 \leq \varphi_n \in C_c^\infty(\mathbb{R})$ such that, restricted to the interval $[0, \|a\|]$, we have $\chi_{(1/n, \|a\|]} \leq \varphi_n \leq \chi_{(1/(n+1), \|a\|]}$. This immediately implies that $P_n \leq \varphi_n(a)$, $\varphi_n^2(a) \leq P_{n+1}$. For the second limiting process we define $a_n^\varphi := a\varphi_n(a)$. Now, since

$$a^\delta (1 - \varphi_n(a)) \leq a^\delta (1 - \varphi_n^2(a)) \leq a^\delta (1 - P_n) = \int_0^{1/n} \lambda^\delta dE_a(\lambda),$$

we have

$$(5.4) \quad \|a^\delta (1 - \varphi_n(a))\| \leq \|a^\delta (1 - \varphi_n^2(a))\| \leq \|a^\delta (1 - P_n)\| \leq n^{-\delta}, \quad \forall \delta > 0.$$

Finally, from $P_n \leq \varphi_n(a) \leq P_{n+1}$, we deduce that

$$(5.5) \quad \frac{1}{n} P_n \leq a_n \leq a_n^\varphi \leq a_{n+1},$$

and that

$$(5.6) \quad \varphi_n(a)P_{n+1} = \varphi_n(a), \quad \varphi_n(a)P_n = P_n.$$

The reason why two approximations are required is as follows. The projection based method allows the use of several operator inequalities, most notably [7, Lemma 3.3 (ii)]. If we were then willing to assume that $[P_n, G] \in Z_1^0$, then the following proof would simplify considerably. However, this assumption is highly implausible in the examples. So we introduce the smooth approximation scheme, and a more complex proof, in order to obtain a result which is actually applicable to the examples.

The following is our main technical result from which Theorem 5.14 will follow easily.

Proposition 5.4. *Let $0 \leq a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then*

$$\lim_{s \searrow 1} (s-1) \|aG^s a - (aGa)^s\|_1 = 0.$$

The proof of the proposition proceeds by writing

$$\begin{aligned} aG^s a - (aGa)^s &= \left[aG^s a - a_n^\varphi G^s a_n^\varphi \right] + \left[a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a \right] \\ &+ \left[a(\varphi_n(a)G\varphi_n(a))^s a - a(P_n G P_n)^s a \right] + \left[a(P_n G P_n)^s a - (a_n G a_n)^s \right] + \left[(a_n G a_n)^s - (aGa)^s \right] \end{aligned}$$

and then controlling each successive difference in this equality in the trace norm. The following sequence of lemmas achieves this goal.

Lemma 5.5. *Let $0 \leq a \in \mathcal{N}$ such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then*

$$\limsup_{s \searrow 1} (s-1) \|aG^s a - a_n^\varphi G^s a_n^\varphi\|_1 \leq \left(n^{-2\delta} + 2\|a\|^\delta n^{-\delta} \right) \|a^{1-\delta}\|_\zeta^2.$$

Proof. Since $0 \leq a_n^{\varphi^2} \leq a^2$ it follows from Lemma 4.7, *ii*), that $a_n^\varphi \in B_\zeta$ and that the function $s \mapsto (s-1)\tau(a_n^\varphi G^s a_n^\varphi)$, for $s \geq 1$, is well defined and bounded. Using the equality

$$a_n^\varphi G^s a_n^\varphi - aG^s a = (1 - \varphi_n(a))aG^s a(1 - \varphi_n(a)) - aG^s a(1 - \varphi_n(a)) - (1 - \varphi_n(a))aG^s a,$$

we obtain

$$\|aG^s a - a_n^\varphi G^s a_n^\varphi\|_1 \leq \left(\|(1 - \varphi_n(a))a^\delta\|^2 + 2\|(1 - \varphi_n(a))a^\delta\| \|a^\delta\| \right) \|a^{1-\delta} G^s a^{1-\delta}\|_1.$$

Since $\|a^\delta(1 - \varphi_n(a))\| \leq n^{-\delta}$, by equation (5.4), we immediately deduce the result. \square

Lemma 5.6. *Let $0 \leq a \in \mathcal{N}$ such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then there exists two constants $C_1, C_2 > 0$, uniform in n , such that*

$$\begin{aligned} i) \quad & \limsup_{s \searrow 1} (s-1) \|(a_{n+1} G a_{n+1})^s - (a_n G a_n)^s\|_1 \leq C_1 n^{-\delta/2}, \\ ii) \quad & \limsup_{s \searrow 1} (s-1) \|(a_n G a_n)^s - (aGa)^s\|_1 \leq C_2 n^{-\delta/2}. \end{aligned}$$

Proof. To prove *i)*, let $A_n := a_{n+1} G a_{n+1}$ and $B_n := a_n G a_n$. Then

$$\begin{aligned} \|A_n^s - B_n^s\|_1 &= \|A_n^{s/2}(A_n^{s/2} - B_n^{s/2}) - (A_n^{s/2} - B_n^{s/2})B_n^{s/2}\|_1 \leq (\|A_n^{s/2}\|_2 + \|B_n^{s/2}\|_2) \|A_n^{s/2} - B_n^{s/2}\|_2 \\ &\leq (\|A_n\|_s^{s/2} + \|B_n\|_s^{s/2}) \|A_n - B_n\|_s^{s/2}, \end{aligned}$$

by the BKS inequality since $0 < s/2 < 1$. Then we use $\|A_n\|_s \leq \|aGa\|_s$ and $\|B_n\|_s \leq \|aGa\|_s$, together with

$$\|A_n - B_n\|_s = \|a_{n+1}Ga(P_{n+1} - P_n) - (P_n - P_{n+1})aGa_n\|_s \leq 2 \|a^\delta(P_{n+1} - P_n)\| \|a^{1-\delta}Ga\|_s,$$

to obtain

$$\|A_n^s - B_n^s\|_1 \leq 2^{s/2+1} \|a\|^{3\delta s/2} \|a^\delta(P_{n+1} - P_n)\|^{s/2} \|a^{1-\delta}Ga\|_s^{s/2}.$$

This concludes the proof since $(s-1)\|a^{1-\delta}Ga\|_s^{s/2}$ is bounded and $\|a^\delta(P_{n+1} - P_n)\|^{s/2} \leq n^{-s\delta/2}$.

To prove *ii)*, one uses the same strategy applied to $A_n = a_nGa_n$ and $B_n = aGa$. \square

The following result is strongly inspired by [7, Lemmas 3.3-3.5]:

Lemma 5.7. *Let $P \in \mathcal{N}$ be a projector and $0 \leq a \in B_\zeta$ such that $[a, P] = 0$ and $a \geq mP$, for some $m \in (0, 1)$. Then*

$$\lim_{s \searrow 1} (s-1) \|a(PGP)^s a - (aPGPa)^s\|_1 = 0.$$

Proof. By [7, Lemma 3.3 i)], we have

$$(aPGPa)^s \leq \|a\|^{2(s-1)} a(PGP)^s a,$$

The result follows if we can show that

$$(5.7) \quad (aPGPa)^s \geq m^{2(s-1)} a(PGP)^s a,$$

as we would then have

$$(5.8) \quad (m^{2(s-1)} - 1) a(PGP)^s a \leq (aPGPa)^s - a(PGP)^s a \leq (\|a\|^{2(s-1)} - 1) a(PGP)^s a,$$

and this suffices by the following reasoning. If $\|a\| \leq 1$, then

$$0 \leq a(PGP)^s a - (aPGPa)^s \leq (1 - m^{2(s-1)}) a(PGP)^s a,$$

and the claim follows. So assume $\|a\| > 1$. Then,

$$-(\|a\|^{2(s-1)} - 1) a(PGP)^s a \leq a(PGP)^s a - (aPGPa)^s \leq (1 - m^{2(s-1)}) a(PGP)^s a.$$

Setting

$$0 \leq b = (\|a\|^{2(s-1)} - 1), \quad 0 \leq c = (1 - m^{2(s-1)}), \quad A = a(PGP)^s a, \quad X = a(PGP)^s a - (aPGPa)^s,$$

we have $0 \leq X + bA \leq (c + b)A$, and thus

$$\|X\|_1 \leq \|X + bA\|_1 + b\|A\|_1 \leq (c + 2b)\|A\|_1,$$

that is

$$\|a(PGP)^s a - (aPGPa)^s\|_1 \leq \left((1 - m^{2(s-1)}) + 2(\|a\|^{2(s-1)} - 1) \right) \|a(PGP)^s a\|_1,$$

which gives the result since

$$\|a(PGP)^s a\|_1 \leq \|aPG^s Pa\|_1 = \|PaG^s aP\|_1 \leq \|aG^s a\|_1,$$

where we have used the operator inequality $a(PGP)^s a \leq aPG^s Pa$, from [7, Lemma 3.3 i)].

To prove (5.7), decompose \mathcal{H} as $P\mathcal{H} \oplus (1 - P)\mathcal{H}$. Since $[P, a] = 0$, we know that

$$(aPGPa)^s = P(aPGPa)^s P, \quad \text{and} \quad a(PGP)^s a = Pa(PGP)^s aP,$$

and so their restrictions to $(P\mathcal{H})_n^\perp$ are zero and so (5.7) holds on $(1 - P)\mathcal{H}$. Since $a \geq mP$, its restriction to $P\mathcal{H}$ is an invertible element of $P\mathcal{N}P$ and [7, Lemma 3.3 ii)] gives the result. \square

Next we prove some results involving both the projectors P_n and their smooth versions $\varphi_n(a)$.

Lemma 5.8. *Let $0 \leq a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then there exists $C > 0$, uniform in n , such that*

$$\limsup_{s \searrow 1} (s - 1) \|a(P_nGP_n)^s a - a(P_{n+1}GP_{n+1})^s a\|_1 \leq C n^{-\delta/2}.$$

Proof. Write

$$\begin{aligned} a(P_nGP_n)^s a - a(P_{n+1}GP_{n+1})^s a &= (a(P_nGP_n)^s a - (a_nGa_n)^s) + ((a_nGa_n)^s - (a_{n+1}Ga_{n+1})^s) \\ &\quad + ((a_{n+1}Ga_{n+1})^s - a(P_{n+1}GP_{n+1})^s a), \end{aligned}$$

and apply Lemma 5.6 and Lemma 5.7. \square

Lemma 5.9. *Let $0 \leq a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then there exists $C > 0$, uniform in n , such that*

$$\limsup_{s \searrow 1} (s - 1) \|a(\varphi_n(a)G\varphi_n(a))^s a - a(P_nGP_n)^s a\|_1 \leq C n^{-\delta/2}.$$

Proof. By equation (5.6), we have $\varphi_n(a) = P_{n+1}\varphi_n(a)$, while $P_n = \varphi_n(a)P_n$. Thus,

$$\begin{aligned} a(\varphi_n(a)G\varphi_n(a))^s a - a(P_nGP_n)^s a &= a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a - a(P_n\varphi_n(a)G\varphi_n(a)P_n)^s a \\ &= \left[a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a - (aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a)^s \right] \\ &\quad + \left[(aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a)^s - (aP_n\varphi_n(a)G\varphi_n(a)P_na)^s \right] \\ &\quad + \left[(aP_n\varphi_n(a)G\varphi_n(a)P_na)^s - a(P_n\varphi_n(a)G\varphi_n(a)P_n)^s a \right]. \end{aligned}$$

For the first term in parentheses, we can apply Lemma 5.7, with the modification that we replace G there by $\varphi_n(a)G\varphi_n(a)$, to obtain a vanishing contribution. Indeed, following line by line the proof of Lemma 5.7 with the indicated modification, we get the operator inequalities

$$\begin{aligned} (m^{2(s-1)} - 1)a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \\ \leq (aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a)^s - a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \\ \leq (\|a\|^{2(s-1)} - 1)a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a. \end{aligned}$$

Combining these operator inequalities with $a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \leq aG^s a$, we obtain

$$\lim_{s \searrow 1} (s-1) \|a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a - (aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a)^s\|_1 = 0.$$

Replacing P_{n+1} by P_n gives the same conclusion for the last term in parentheses.

For the middle term, we can apply Lemma 5.8 with the replacement $a \mapsto a_n^\varphi$, to obtain the desired trace-norm bound. Indeed, since $a_n^{\varphi^2} = a^2\varphi_n^2(a) \leq a^2$, we infer from Lemma 4.7 ii) that $a_n^\varphi \in B_\zeta$ and since $(a_n^\varphi)^{1-\delta} = a^{1-\delta}\varphi_n(a)^{1-\delta} \leq a^{1-\delta}$, we see that $(a_n^\varphi)^{1-\delta}$ belongs to B_ζ too. \square

To control the trace-norm of $a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a$, we need the following identity.

Lemma 5.10. *Let $0 \leq A, B \in \mathcal{N}$ with B injective. Then for all $1 \leq s \leq 2$ we have the equality*

$$(ABA)^s = AB^{1/2}(B^{1/2}A^2B^{1/2})^{s-1}B^{1/2}A.$$

Proof. Let $B^{1/2}A = v|B^{1/2}A|$ so that $AB^{1/2} = |B^{1/2}A|v^*$, where the partial isometry v satisfies

$$v : (\ker(B^{1/2}A))^\perp \rightarrow \overline{\text{range}(B^{1/2}A)}.$$

Moreover,

$$\text{support}(v^*v) = (\ker(B^{1/2}A))^\perp, \quad \text{support}(vv^*) = \overline{\text{range}(B^{1/2}A)}.$$

Since B is injective, we have $\ker(B^{1/2}A) = \ker(A)$, and thus $v^*v = \text{support}(A)$. Next, we remark that

$$\overline{\text{range}(B^{1/2}A)}^\perp = \ker(AB^{1/2}) = \ker(|AB^{1/2}|) = \ker(B^{1/2}A^2B^{1/2}),$$

and thus vv^* is actually the projection onto the orthogonal complement of the kernel of $B^{1/2}A^2B^{1/2}$, so we immediately conclude that

$$(5.9) \quad vv^*B^{1/2}A^2B^{1/2} = B^{1/2}A^2B^{1/2}vv^* = B^{1/2}A^2B^{1/2}.$$

Observing that

$$AB^{1/2} = v^*vAB^{1/2} = v^*v|B^{1/2}A|v^* = v^*B^{1/2}Av^* \quad \text{and}$$

$$B^{1/2}A = B^{1/2}Av^*v = v|B^{1/2}A|v^*v = vAB^{1/2}v,$$

we find

$$ABA = v^*B^{1/2}Av^*vAB^{1/2}v = v^*B^{1/2}A^2B^{1/2}v.$$

Thus restricting ABA to $v^*v\mathcal{H}$ (it is zero on $(1 - v^*v)\mathcal{H}$), we obtain

$$\begin{aligned} (ABA)^{s-1} &= (v^*B^{1/2}A^2B^{1/2}v)^{s-1} \\ &= C_{s-1} \int_0^\infty \lambda^{-s+1} v^*B^{1/2}A^2B^{1/2}v(v^*v + \lambda v^*B^{1/2}A^2B^{1/2}v)^{-1} d\lambda \\ &= C_{s-1} \int_0^\infty \lambda^{-s+1} v^*B^{1/2}A^2B^{1/2}(vv^* + \lambda B^{1/2}A^2B^{1/2})^{-1} d\lambda v \\ &= v^* (B^{1/2}A^2B^{1/2})^{s-1} v, \end{aligned}$$

where we used that vv^* is the support projection of $B^{1/2}A^2B^{1/2}$. We conclude by

$$\begin{aligned} (ABA)^s &= |B^{1/2}A| (ABA)^{s-1} |B^{1/2}A| \\ &= |B^{1/2}A| v^* (B^{1/2}A^2B^{1/2})^{s-1} v |B^{1/2}A| = AB^{1/2} (B^{1/2}A^2B^{1/2})^{s-1} B^{1/2}A. \end{aligned} \quad \square$$

Lemma 5.11. *Let $0 \leq a \in \mathcal{N}$ and suppose that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then there exists an absolute constant $C > 0$ such that*

$$\limsup_{s \searrow 1} (s-1) \|a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a\|_1 \leq C n^{-\delta/2}.$$

Proof. We remark first from Lemma 4.7 i), that $a^{1-\delta} \in B_\zeta$ implies $a \in B_\zeta$ too. Then, as $\varphi_n(a) \leq P_{n+1} \leq (n+1)a_{n+1}$, we readily see that $\varphi_n(a)G\varphi_n(a) \in \mathcal{Z}_1$ and thus $(\varphi_n(a)G\varphi_n(a))^s$ is trace-class for all $s > 1$, so that we are entitled to take the trace norm as in the statement of the lemma.

Lemma 5.10 applied to $A = \varphi_n(a)$ and $B = G$ (which is injective), gives

$$\begin{aligned} a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a &= a_n^\varphi G^{1/2} (G^{s-1} - (G^{1/2}\varphi_n(a)^2 G^{1/2})^{s-1}) G^{1/2} a_n^\varphi \\ &= a_n^\varphi G^{1/2} \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \lambda^{-\varepsilon} \left(G(1 + \lambda G)^{-1} - G^{1/2}\varphi_n(a)^2 G^{1/2}(1 + \lambda G^{1/2}\varphi_n(a)^2 G^{1/2})^{-1} \right) d\lambda G^{1/2} a_n^\varphi \\ &= a_n^\varphi G^{1/2} \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \lambda^{-\varepsilon} (1 + \lambda G)^{-1} G^{1/2} (1 - \varphi_n^2(a)) G^{1/2} (1 + \lambda G^{1/2}\varphi_n(a)^2 G^{1/2})^{-1} d\lambda G^{1/2} a_n^\varphi, \end{aligned}$$

where we have defined $\varepsilon := s - 1$ and we have used in the third equality, the identity

$$A(1 + \lambda A)^{-1} - B(1 + \lambda B)^{-1} = (1 + \lambda A)^{-1} (A - B) (1 + \lambda B)^{-1}, \quad 0 \leq A, B \in \mathcal{N}, \lambda > 0.$$

Hence

$$a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a = \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty X_{\varepsilon,n}(\lambda) \lambda^{-\varepsilon} d\lambda,$$

with

$$X_{\varepsilon,n}(\lambda) = a_n^\varphi G(1 + \lambda G)^{-1} (1 - \varphi_n^2(a)) G^{1/2} (1 + \lambda G^{1/2}\varphi_n(a)^2 G^{1/2})^{-1} G^{1/2} a_n^\varphi.$$

Commuting $1 - \varphi_n^2(a)$ with $G(1 + \lambda G)^{-1}$ on its left, we obtain $X_{\varepsilon,n}(\lambda) = Y_{\varepsilon,n}(\lambda) + Z_{\varepsilon,n}(\lambda)$, with

$$\begin{aligned} Y_{\varepsilon,n}(\lambda) &:= a_n^\varphi (1 - \varphi_n^2(a)) G^{1/2} (G^{-1} + \lambda)^{-1} (1 + \lambda G^{1/2} \varphi_n(a)^2 G^{1/2})^{-1} G^{1/2} a_n^\varphi, \\ Z_{\varepsilon,n}(\lambda) &:= a_n^\varphi (1 + \lambda G)^{-1} [G, \varphi_n^2(a)] (G^{-1} + \lambda)^{-1} \left(1 - \lambda \varphi_n(a)^2 (G^{-1} + \lambda \varphi_n(a)^2)^{-1} \right) a_n^\varphi, \end{aligned}$$

where we have used the relations

$$[G(1 + \lambda G)^{-1}, (1 - \varphi_n^2(a))] = -\lambda^{-1} [(1 + \lambda G)^{-1}, \varphi_n^2(a)] = (1 + \lambda G)^{-1} [G, \varphi_n^2(a)] (1 + \lambda G)^{-1},$$

and

$$G^{1/2} (1 + \lambda G^{1/2} \varphi_n(a)^2 G^{1/2})^{-1} G^{1/2} = (G^{-1} + \lambda \varphi_n(a)^2)^{-1} = G(1 - \lambda \varphi_n(a)^2 (G^{-1} + \lambda \varphi_n(a)^2)^{-1}),$$

from the identity

$$(A + B)^{-1} = A^{-1} (1 - B(A + B)^{-1}), \quad A \geq 1, B \geq 0.$$

For $\|Y_{\varepsilon,n}(\lambda)\|_1$, we use the Hölder inequality to obtain the upper bound

$$\begin{aligned} \|a^{\delta/2} (1 - \varphi_n^2(a))\| \|a^{1-\delta/2} G^{(1+\varepsilon)/2}\|_{\frac{2+\varepsilon}{1+\varepsilon}} \|G^{-\varepsilon/2} (G^{-1} + \lambda)^{-1}\| \| (1 + \lambda G^{1/2} \varphi_n(a)^2 G^{1/2})^{-1} \| \|G^{1/2} a\|_{2+\varepsilon} \\ \leq n^{-\delta/2} (1 + \lambda)^{-1+\varepsilon/2} \|a^{1-\delta/2} G^{(1+\varepsilon)/2}\|_{\frac{2+\varepsilon}{1+\varepsilon}} \|G^{1/2} a\|_{2+\varepsilon}, \end{aligned}$$

where we have used equation (5.4) and

$$G^{-\varepsilon/2} (G^{-1} + \lambda)^{-1} \leq (G^{-1} + \lambda)^{-1+\varepsilon/2} \leq (1 + \lambda)^{-1+\varepsilon/2}.$$

Next, from the operator inequality, [7, Lemma 3.3 i)], $(aGa)^{1+\varepsilon/2} \leq \|a\|^\varepsilon a G^{1+\varepsilon/2} a$, we obtain

$$\begin{aligned} \|G^{1/2} a\|_{2+\varepsilon} &= \|aGa\|_{1+\varepsilon/2}^{1/2} \\ (5.10) \quad &\leq \|a\|^{\varepsilon/(2+\varepsilon)} \|aG^{1+\varepsilon/2} a\|_1^{(2+\varepsilon)^{-1}} \leq \|a\|^{\varepsilon/(2+\varepsilon)} \|a\|_\zeta^{2/(2+\varepsilon)} (2/\varepsilon)^{(2+\varepsilon)^{-1}} \leq C \varepsilon^{-1/2}. \end{aligned}$$

To evaluate $\|a^{1-\delta/2} G^{(1+\varepsilon)/2}\|_{(2+\varepsilon)/(1+\varepsilon)}$, we write

$$a^{1-\delta/2} G^{(1+\varepsilon)/2} = a^{\delta/2} G^{(1+\varepsilon)/2} a^{1-\delta} + a^{\delta/2} [a^{1-\delta}, G^{(1+\varepsilon)/2}].$$

For the first term, we obtain

$$\|a^{\delta/2} G^{(1+\varepsilon)/2} a^{1-\delta}\|_{(2+\varepsilon)/(1+\varepsilon)} \leq \|a^{\delta/2} G^{\varepsilon/2}\|_{1+2/\varepsilon} \|G^{1/2} a^{1-\delta}\|_{2+\varepsilon} \leq C \varepsilon^{-1/2},$$

where we used that $\|a^{\delta/2} G^{\varepsilon/2}\|_{1+2/\varepsilon}$ remains bounded when $\varepsilon \rightarrow 0^+$, from Lemma 5.3 ii), and the estimate of equation (5.10) for the second part, since $a^{1-\delta} \in B_\zeta$ by assumption. It remains to treat the commutator term, for which Lemma 5.1 gives us

$$\|[a^{1-\delta}, G^{(1+\varepsilon)/2}]\|_{(2+\varepsilon)/(1+\varepsilon)} \leq C \|[a^{1-\delta}, G]\|_{1+\varepsilon/2}^{(1+\varepsilon)/2}.$$

We conclude using Lemma 5.3 i) that $\varepsilon^{1/2} \|[a^{1-\delta}, G^{(1+\varepsilon)/2}]\|_{(2+\varepsilon)/(1+\varepsilon)} \rightarrow 0$ when $\varepsilon \rightarrow 0^+$. Hence, we have shown that

$$\limsup_{\varepsilon \searrow 0} \varepsilon \|Y_{\varepsilon,n}(\lambda)\|_1 \leq C n^{-\delta/2} (1 + \lambda)^{-1+\varepsilon/2}.$$

It remains to treat $Z_{\varepsilon,n}(\lambda)$ which we estimate in trace-norm as

$$\begin{aligned} \|Z_{\varepsilon,n}(\lambda)\|_1 &\leq \|a_n^\varphi(1+\lambda G)^{-1}[G, \varphi_n^2(a)]G(1+\lambda G)^{-1}a_n^\varphi\|_1 \\ &\quad + \|a_n^\varphi(1+\lambda G)^{-1}[G, \varphi_n^2(a)]G(1+\lambda G)^{-1}\lambda\varphi_n^2(a)(G^{-1}+\lambda\varphi_n(a)^2)^{-1}a_n^\varphi\|_1. \end{aligned}$$

We estimate the first term by

$$\begin{aligned} \|a\| \| [G, \varphi_n^2(a)] \|_{1+\varepsilon/2} \|G^{1-\varepsilon/2}(1+\lambda G)^{-1}\| \|G^{\varepsilon/2}a\|_{1+2/\varepsilon} \\ \leq \|a\| \| [G, \varphi_n^2(a)] \|_{1+\varepsilon/2} \|G^{\varepsilon/2}a\|_{1+2/\varepsilon} (1+\lambda)^{-1+\varepsilon/2}. \end{aligned}$$

For the second term, we obtain the bound

$$\|a\|^2 \| [G, \varphi_n^2(a)] \|_{1+\varepsilon/2} \|G^{\varepsilon/2}\varphi_n(a)\|_{1+2/\varepsilon} \left\| \varphi_n(a) \frac{\lambda}{G^{-1}+\lambda\varphi_n(a)^2} \varphi_n(a) \right\| (1+\lambda)^{-1+\varepsilon/2}.$$

Since $G^{-1} \geq 1$, we have the estimate

$$(G^{-1} + \lambda\varphi_n(a)^2)^{-1} \leq (1 + \lambda\varphi_n(a)^2)^{-1},$$

and thus

$$\|\varphi_n(a)\lambda(G^{-1} + \lambda\varphi_n(a)^2)^{-1}\varphi_n(a)\| \leq \|\lambda\varphi_n^2(a)(1 + \lambda\varphi_n(a)^2)^{-1}\| \leq 1.$$

Using $\varphi_n(a) \leq n a$, we obtain $\|G^{\varepsilon/2}\varphi_n(a)\|_{1+2/\varepsilon} \leq n\|G^{\varepsilon/2}a\|_{1+2/\varepsilon}$ and so

$$\begin{aligned} \|Z_{\varepsilon,\lambda}^n\|_1 &\leq C(1+n) \| [G, \varphi_n^2(a)] \|_{1+\varepsilon/2} \|G^{\varepsilon/2}a\|_{1+2/\varepsilon} (1+\lambda)^{-1+\varepsilon/2} \\ &\leq C(1+n) \|\widehat{\varphi_n^{2'}}\|_1 \| [G, a] \|_{1+\varepsilon/2} \|G^{\varepsilon/2}a\|_{1+2/\varepsilon} (1+\lambda)^{-1+\varepsilon/2}. \end{aligned}$$

We have used Lemma 5.2 to obtain the last inequality. We stress that $\|\widehat{\varphi_n^{2'}}\|_1$ is not uniform in n since φ_n^2 pointwise-converges to a step function. However, combining Theorem 3.1 from [30] with Theorem 4 from [31], and taking into account that \mathcal{Z}_1^0 is an interpolation space for the couple $(\mathcal{L}^1, \mathcal{N})$, we get from $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$ that $[G, a] \in \mathcal{Z}_1^0$ as well. Thus, by Lemma 5.3 i), ii), we know that $\varepsilon\|[G, a]\|_{1+\varepsilon} \rightarrow 0$, while $\|G^\varepsilon a\|_{1+1/\varepsilon}$ remains bounded when $\varepsilon \rightarrow 0^+$. Putting everything together, we obtain the announced result:

$$\begin{aligned} \limsup_{s \searrow 1} (s-1) \|a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a\|_1 \\ \leq \limsup_{\varepsilon \searrow 0} \varepsilon \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty (\|Y_{\varepsilon,n}(\lambda)\|_1 + \|Z_{\varepsilon,n}(\lambda)\|_1) \lambda^{-\varepsilon} d\lambda \leq Cn^{-\delta/2}. \quad \square \end{aligned}$$

We are now ready to complete the proof of our main technical result.

Proof of Proposition 5.4. We write:

$$\begin{aligned} aG^s a - (aGa)^s &= \left[aG^s a - a_n^\varphi G^s a_n^\varphi \right] + \left[a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a \right] \\ &\quad + \left[a(\varphi_n(a)G\varphi_n(a))^s a - a(P_n G P_n)^s a \right] + \left[a(P_n G P_n)^s a - (a_n G a_n)^s \right] + \left[(a_n G a_n)^s - (aGa)^s \right]. \end{aligned}$$

The \limsup , $s \rightarrow 1^+$ of the trace norm of the first bracket multiplied by $(s-1)$, is bounded by $n^{-\delta}$ by Lemma 5.5, the second is bounded by $n^{-\delta/2}$ by Lemma 5.11, the third is bounded by $n^{-\delta/2}$ by Lemma 5.9, the fourth is bounded by 0 by Lemma 5.7 and the fifth by $n^{-\delta/2}$ by Lemma 5.6 ii). This concludes the proof since it implies: $\limsup_{s \rightarrow 1^+} \|aG^s a - (aGa)^s\|_1 \leq C n^{-\delta/2}$, $\forall n \in \mathbb{N}$. \square

Proposition 5.4 immediately gives us

Corollary 5.12. *Assume that $0 \leq a \in \mathcal{N}$ is such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then*

(i) $\lim_{s \rightarrow 1^+} (s-1)\tau(aG^s a)$ exists if and only if $\lim_{s \rightarrow 1^+} (s-1)\tau((aGa)^s)$ exists and then they are equal;

(ii) More generally, for any Banach limits ω , we have

$$\tilde{\omega} - \lim_{s \searrow 1} (s-1)\tau(aG^s a) = \tilde{\omega} - \lim_{s \searrow 1} (s-1)\tau((aGa)^s).$$

5.3. Dixmier-traces computation. We have arrived at the following analogue of [7, Corollary 3.7].

Proposition 5.13. *Let $0 \leq a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then, if any one of the following limits exist they all do and all coincide*

(1) $\lim_{t \rightarrow \infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(aGa) ds,$

(2) $\lim_{r \rightarrow \infty} \frac{1}{r} \tau((aGa)^{1+\frac{1}{r}}),$

(3) $\lim_{r \rightarrow \infty} \frac{1}{r} \zeta(a, a; 1 + \frac{1}{r}),$

(4) $\lim_{\lambda \rightarrow \infty} (Mg(a, a; \cdot))(\lambda).$

Furthermore, the existence of any of the above limits is equivalent to

(5) every generalized limit ω which is dilation invariant yields the same value $\tau_\omega(aGa)$ and the latter value coincides with the value of the limits above.

Proof. The simultaneous existence and equality of (2) and (3) (resp. (3) and (4)) follows from Corollary 5.12 (resp. Theorem 4.6). Recall now that the assumption $a \in B_\zeta$ guarantees $aGa \in \mathcal{Z}_1$. The assertion “(2) exists if and only if (1) exists and then they are equal” is known, it follows e.g. by the same argument as in the proof of [7, Corollary 3.7] or by the argument given at the beginning of the proof of [1, Theorem 2]. If (1) exists then it equals $\tau_\omega(a^*Ga)$ by definition. The equivalence of (4) and the existence of the limit (1) and their equality follows from the main result of [24]. \square

We have arrived at the following nonunital analogue of [7, Theorem 3.8], [8, Theorem 4.11] and [23, Corollary 3.3].

Theorem 5.14. *Assume that $a \in \mathcal{N}$ is self adjoint and let $a = a_+ - a_-$ be the decomposition into the difference of nonnegative operators. Assume that there exists $\delta > 0$ with $a_\pm^{1/2-\delta} \in B_\zeta$, and $[G, a_\pm^{1/2-\delta}] \in \mathcal{Z}_1^0$. Then $aG \in \mathcal{Z}_1$, and moreover,*

(i) if $\lim_{s \rightarrow 1^+} (s-1)(\zeta(a_+^{1/2}, a_+^{1/2}; s) - \zeta(a_-^{1/2}, a_-^{1/2}; s))$ exists, then it is equal to $\tau_\omega(aG)$ where we choose ω as in the proof of [8, Theorem 4.11],

(ii) more generally, if we choose functionals ω and $\tilde{\omega}$ as in the proof of [8, Theorem 4.11], then

$$\tau_\omega(aG) = \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} (\zeta(a_+^{1/2}, a_+^{1/2}; 1 + \frac{1}{r}) - \zeta(a_-^{1/2}, a_-^{1/2}; 1 + \frac{1}{r})).$$

Proof. As observed earlier, recall that $a_\pm^{1/2-\delta} \in B_\zeta$ implies $a_\pm^{1/2} \in B_\zeta$ and $[G, a_\pm^{1/2-\delta}] \in \mathcal{Z}_1^0$ implies $[G, a_\pm^{1/2}] \in \mathcal{Z}_1^0$. Thus

$$aG = a_+G - a_-G = a_+^{1/2}Ga_+^{1/2} - a_-^{1/2}Ga_-^{1/2} + a_+^{1/2}[a_+^{1/2}, G] - a_-^{1/2}[a_-^{1/2}, G] \in \mathcal{Z}_1,$$

with $\tau_\omega(aG) = \tau_\omega(a_+^{1/2}Ga_+^{1/2}) - \tau_\omega(a_-^{1/2}Ga_-^{1/2})$, where ω (and latter $\tilde{\omega}$) has been chosen as in the proof of [8, Theorem 4.11]. We only need to prove part ii); part i) will then follow from general facts on Banach limits. By [7, Lemma 3.2(i)], [8, Theorem 4.11], Proposition 5.12 and the remark above, we have

$$\begin{aligned} \tau_\omega(aG) &= \tau_\omega(a_+^{1/2}Ga_+^{1/2}) - \tau_\omega(a_-^{1/2}Ga_-^{1/2}) = \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} (\tau((a_+^{1/2}Ga_+^{1/2})^{1+\frac{1}{r}}) - \tau((a_-^{1/2}Ga_-^{1/2})^{1+\frac{1}{r}})) \\ &= \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} (\tau(a_+^{1/2}G^{1+\frac{1}{r}}a_+^{1/2}) - \tau(a_-^{1/2}G^{1+\frac{1}{r}}a_-^{1/2})), \end{aligned}$$

which concludes the proof. \square

By considering independently real and imaginary parts of $a \in B_\zeta$, we get an analogous result for non-self-adjoint elements. Moreover, we could have stated a similar result using the Césaro mean of the heat-trace function instead of the zeta function. Namely, under the same assumptions as those of Theorem 5.14, it is true that

$$\tau_\omega(aG) = \omega - \lim_{\lambda \rightarrow \infty} ((Mg(a_+^{1/2}, a_+^{1/2}; \cdot))(\lambda) - (Mg(a_-^{1/2}, a_-^{1/2}; \cdot))(\lambda)).$$

We close this Section with

Proposition 5.15. *Let $a \in B_\zeta$ be such that $[G, a] \in \mathcal{Z}_1^0$. Then for any Dixmier trace*

$$0 \leq \tau_\omega(a^*aG) = \tau_\omega(aa^*G).$$

Proof. By Theorem 4.8, $a^*Ga, aGa^* \in \mathcal{Z}_1$, and since $aa^*G = aGa^* + a[a^*, G]$, $a^*aG = a^*Ga + a^*[a, G]$, aa^*G and a^*aG belong to \mathcal{Z}_1 as well.

Now, for $A \in \mathcal{Z}_2$, we have

$$\tau_\omega(A^*A) = \omega - \lim_{t \rightarrow +\infty} \frac{\int_0^t \mu_s(A)^2 ds}{\log(1+t)} = \omega - \lim_{t \rightarrow +\infty} \frac{\int_0^t \mu_s(A^*)^2 ds}{\log(1+t)} = \tau_\omega(AA^*).$$

Now we do some rearranging

$$\begin{aligned}\tau_\omega(a^*aG) &= \tau_\omega(a^*Ga) + \tau_\omega(a^*[a, G]) = \tau_\omega(a^*Ga) = \tau_\omega(G^{1/2}aa^*G^{1/2}) \\ &= \tau_\omega(aG^{1/2}a^*G^{1/2}) + \tau_\omega([G^{1/2}, a]a^*G^{1/2}) = \tau_\omega(aa^*G) + \tau_\omega(a[G^{1/2}, a^*]G^{1/2}),\end{aligned}$$

as the Dixmier trace vanishes on the ideal \mathcal{Z}_1^0 , by Lemma 5.1 $[G^{1/2}, a] \in \mathcal{Z}_2^0$, and the fact that $\mathcal{Z}_2^0 \mathcal{Z}_2 \subset \mathcal{Z}_1^0$. We complete the proof by observing that

$$a[G^{1/2}, a^*]G^{1/2} = a[G, a^*] - aG^{1/2}[G^{1/2}, a^*],$$

and that both terms on the right hand side of this last equation are in \mathcal{Z}_1^0 . \square

6. THE CONVERSE ESTIMATE

In the previous Section we have shown that $a \in B_\zeta \Rightarrow a^*Ga \in \mathcal{Z}_1$. But the converse requires more assumptions. We demonstrate this by exhibiting a counter-example. Let us introduce one more Banach *-sub-algebra of \mathcal{N} :

$$B_{\mathcal{Z}_1} = B_{\mathcal{Z}_1}(G) := \{a \in \mathcal{N} : \|a^*Ga\|_{1,\infty} + \|aGa^*\|_{1,\infty} + \|a\|^2 < \infty\}.$$

It is easy to see that in general $a \in B_{\mathcal{Z}_1} \not\Rightarrow a \in B_\zeta$ by considering the case $G = Id_{\mathcal{N}}$. Then $B_{\mathcal{Z}_1} = \mathcal{Z}_2$ and $B_\zeta = \mathcal{L}^2$. More realistic examples from spectral triples are also easy to produce. A positive result in the converse direction is the following.

Proposition 6.1. *Let $0 \leq a \in \mathcal{N}$, be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_{\mathcal{Z}_1}$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then $a \in B_\zeta$.*

The proof of this ‘converse’ estimate relies essentially on the following lemma.

Lemma 6.2. *Let $0 \leq a \in B_{\mathcal{Z}_1}$ and $\delta \in (0, 1)$. Then, the map $\varepsilon \in (0, \delta/2] \mapsto \|a^\delta G^\varepsilon\|_{1+1/\varepsilon}$, is bounded.*

Proof. We know by assumption that $aGa \in \mathcal{Z}_1$ and thus from Proposition 4.9, we know that for $\delta \in (0, 1)$, $a^\delta G^\varepsilon \in \mathcal{Z}_{1/\varepsilon}$ with

$$\|a^\delta G^\varepsilon\|_{1/\varepsilon, \infty} \leq \|a\|^{\delta-2\varepsilon} \|aGa\|_{1,\infty}^\varepsilon.$$

We conclude using the same chain of estimates as in the proof of Lemma 5.3 ii):

$$\begin{aligned}\|a^\delta G^\varepsilon\|_{1+1/\varepsilon} &= \| |a^\delta G^\varepsilon|^{1/\varepsilon} \|_{1+\varepsilon}^\varepsilon \leq \left(\varepsilon^{-\frac{1}{1+\varepsilon}} \| |a^\delta G^\varepsilon|^{1/\varepsilon} \|_{1,\infty} \right)^\varepsilon \\ &= \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|a^\delta G^\varepsilon\|_{1/\varepsilon, \infty} \leq \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \|a\|^{\delta-2\varepsilon} \|aGa\|_{1,\infty}^\varepsilon. \quad \square\end{aligned}$$

Proof of Proposition 6.1. Note that by the stability of B_ζ and $B_{\mathcal{Z}_1}$ under the map $a \mapsto |a|$, we may assume $a \geq 0$. Then, we write

$$aG^{1+\varepsilon}a = a^\delta G^{1/2+\varepsilon} a^{1-\delta} G^{1/2} a + a^\delta [a^{1-\delta}, G^{1/2+\varepsilon}] G^{1/2} a,$$

and thus

$$(6.1) \quad \|aG^{1+\varepsilon}a\|_1 \leq \|a^\delta G^\varepsilon\|_{1+1/\varepsilon} \|G^{1/2}a^{1-\delta}\|_{2+2\varepsilon} \|G^{1/2}a\|_{2+2\varepsilon} \\ + \|a\|^\delta \|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)} \|G^{1/2}a\|_{2+2\varepsilon}.$$

From Lemma 5.1, we have for $\varepsilon \leq 1/2$

$$(6.2) \quad \|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)} \leq \|a\|^{(1-\delta)(1/2-\varepsilon)} \|[a^{1-\delta}, G]\|_{1+\varepsilon}^{1/2+\varepsilon}.$$

Since $[a^{1-\delta}, G] \in \mathcal{Z}_1^0$, we know from Lemma 5.3 i) that $\|[a^{1-\delta}, G]\|_{1+\varepsilon} = o(\varepsilon^{-1})$. In particular, $\|[a^{1-\delta}, G]\|_{1+\varepsilon}^\varepsilon = O(1)$ and thus the inequality (6.2) gives $\|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)} = o(\varepsilon^{-1/2})$. Moreover, since $a^{1-\delta}Ga^{1-\delta} \in \mathcal{Z}_1$, we know from Theorem 2.1 that $\|a^{1-\delta}Ga^{1-\delta}\|_{1+\varepsilon} = O(\varepsilon^{-1})$, which gives

$$\|G^{1/2}a\|_{2+2\varepsilon} \leq \|a\|^\delta \|G^{1/2}a^{1-\delta}\|_{2+2\varepsilon} = \|a\|^\delta \|a^{1-\delta}Ga^{1-\delta}\|_{1+1\varepsilon}^{1/2} = O(\varepsilon^{-1/2}).$$

Finally, by Lemma 6.2 we know that $\|a^\delta G^\varepsilon\|_{1+1/\varepsilon} = O(1)$. Putting everything together, the inequality (6.1) gives us $\|aG^{1+\varepsilon}a\|_1 = O(\varepsilon^{-1})$, that is $\sup_{\varepsilon>0} \varepsilon \|aG^{1+\varepsilon}a\|_1 < \infty$, i.e. $a \in B_\zeta$. \square

Now we can state a \mathcal{Z}_1 version of Proposition 5.4.

Proposition 6.3. *Let $0 \leq a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_{\mathcal{Z}_1}$, and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then*

$$\lim_{s \rightarrow 1^+} (s-1) \|aG^s a - (aGa)^s\|_1 = 0.$$

Proof. From Proposition 6.1, we deduce that $a^{1-\delta/2} \in B_\zeta$. Combining Theorem 3.1 from [30] with Theorem 4 from [31], and taking into account that \mathcal{Z}_1^0 is an interpolation space for the couple $(\mathcal{L}^1, \mathcal{N})$, we get from $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$ that $[G, a] \in \mathcal{Z}_1^0$ as well. Then the claim follows directly from Proposition 5.4. \square

7. NONUNITAL SPECTRAL TRIPLES

We will now use the results of the previous Sections to give an *a posteriori* definition of a finitely summable nonunital semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to (\mathcal{N}, τ) . A semifinite spectral triple consists of a separable Hilbert space \mathcal{H} carrying a faithful representation of \mathcal{N} , together with an essentially self-adjoint operator \mathcal{D} affiliated with \mathcal{N} and a (nonunital) $*$ -sub-algebra \mathcal{A} of \mathcal{N} such that $[\mathcal{D}, a]$ is bounded for all $a \in \mathcal{A}$ and $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$.

The main difference between the notions of finitely summable *unital* and *nonunital* spectral triple is that, in the unital case, \mathcal{D} alone is enough to characterise the spectral dimension. However, the situation in the nonunital case is far more subtle since one needs a delicate interplay between \mathcal{A} and \mathcal{D} to obtain a good definition of spectral dimension.

Despite our previous focus on ‘ L^1 as the square of L^2 ’, we now define summability for spectral triples in an ‘ L^1 ’ fashion. The reason for this is the local index formula, which we will address

elsewhere. However here we will quickly return, via the results of previous sections, to the L^2 type description.

Definition 7.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a nonunital semifinite spectral triple, relative to (\mathcal{N}, τ) . We then let*

$$p := \inf\{s > 0 : \text{for all } 0 \leq a \in \mathcal{A}, \tau(a(1 + \mathcal{D}^2)^{-s/2}) < \infty\},$$

and when it exists, we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable and call p the spectral dimension of the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Remark. This definition is closer in spirit to that employed in the local index formula, and we will examine, and augment, this definition in another place. The use of the unsymmetrised form of the condition amounts to making the strongest possible assumption. We stress the dependence on the algebra \mathcal{A} in the previous definition. Finally, note that by [3] we obtain a positive functional for each $s > 0$ since for $a \geq 0$,

$$\tau(a(1 + \mathcal{D}^2)^{-s/2}) = \tau((1 + \mathcal{D}^2)^{-s/4}a(1 + \mathcal{D}^2)^{-s/4}) \geq 0.$$

Definition 7.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable nonunital semifinite spectral triple, relative to (\mathcal{N}, τ) , with spectral dimension $p \geq 1$. If for all $a \in \mathcal{A}$*

$$\limsup_{s \searrow p} |(s - p)\tau(a(1 + \mathcal{D}^2)^{-s/2})| < \infty$$

we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is \mathcal{Z}_p -summable. In this case we let $B_p := B_\zeta((1 + \mathcal{D}^2)^{-p/2})$.

The special case described by the definition of \mathcal{Z}_p is the nonunital analogue of the most studied summability criteria in the unital case, usually called (p, ∞) -summability. Variations on the definition are certainly possible, and only further examples can determine the best form of the definition. Note that given $(\mathcal{N}, \tau, \mathcal{D})$, we have the inclusions $B_p \subset B_q$, $p \leq q$.

Lemma 7.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a \mathcal{Z}_p -summable nonunital semifinite spectral triple, relative to (\mathcal{N}, τ) . Suppose that there is a $*$ -algebra $\mathcal{B} \subseteq B_p$ such that $\mathcal{A} \subset \mathcal{B}^2 = \text{span}\{b_1 b_2 : b_1, b_2 \in \mathcal{B}\}$ and $b_1[b_2, (1 + \mathcal{D}^2)^{-p/2}] \in \mathcal{Z}_1^0$ for all $b_1, b_2 \in \mathcal{B}$. Then*

$$a \mapsto \tau_\omega(a(1 + \mathcal{D}^2)^{-p/2})$$

defines a positive trace on \mathcal{A} .

Proof. We make our usual abbreviation $G = (1 + \mathcal{D}^2)^{-p/2}$. With the assumption above we have a finite sum $a = \sum_i b_1^i b_2^i$ and so

$$aG = \sum_i b_1^i b_2^i G = \sum_i b_1^i [b_2^i, G] + b_1^i G b_2^i.$$

Since for each i , $b_1^i [b_2^i, G] \in \mathcal{Z}_1^0$ and $b_1^i G b_2^i \in \mathcal{Z}_1$, for any Dixmier trace τ_ω ,

$$\tau_\omega(aG) = \sum_i \tau_\omega(b_1^i G b_2^i)$$

is well-defined. To see that $a \mapsto \tau_\omega(aG)$ is an (unbounded) trace we employ Proposition 5.15. We just need to check that $[a, G] \in Z_1^0$ for positive $a \in \mathcal{A}$. However, using the Leibnitz rule

$$[a, G] = \sum_i b_1^i [b_2^i, G] + [b_1^i, G] b_2^i \in Z_1^0.$$

Positivity follows from [3] again since $\tau_\omega(aG) = \tau_\omega(G^{1/2}aG^{1/2}) \geq 0$ for $a \geq 0$. \square

Corollary 7.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a \mathcal{Z}_p -summable nonunital semifinite spectral triple, relative to (\mathcal{N}, τ) . Suppose that $0 \leq a \in \mathcal{A}$ is of the form b^2 for $b \geq 0$ with $b^{1-\delta} \in \mathcal{B} \subseteq B_p$ for some $\delta > 0$, and with $b[b, G] \in Z_1^0$. Then choosing $\omega, \tilde{\omega}$ as in [8, Theorem 4.11] we have*

$$\tau_\omega(aG) = \tilde{\omega} - \lim_r \frac{1}{r} \tau(bG^{1+\frac{1}{r}}b)$$

Proof. This is the content of our main result Theorem 5.14. \square

Our final aim is to check the validity of our assumptions in the context of spectral triples, using smoothness of the spectral triple.

Lemma 7.5. *In the definition of B_p we may use $G = (1 + \mathcal{D}^2)^{-p/2}$ interchangeably with $G_1 = (1 + |\mathcal{D}|)^{-p}$.*

Proof. First note that we may apply Lemma 4.7 in this situation. This is because we may write $G_1 = Gf(G)$ or $G = G_1g(G_1)$ with $f, g \in L^\infty(\mathbb{R})$. Thus in the definition of B_p either G or G_1 may be used. We note in addition that for the purpose of studying commutators, we may also use G or G_1 interchangeably as can be seen by the following sketch argument. For $a, b \in B_p$ we have $a(G - G_1)b = aG(1 - G_1G^{-1})b$. It is not difficult to find the bounded function f with $G_1 = f(G)$. Now define a function h such that $1 - f(G)G^{-1} = G^{1/p}h(G)$. Then some careful calculation proves explicitly that h is bounded. So we have $a(G - G_1)b = aG^{1/2}h(G)G^{1/2+1/p}b$ which is trace class. \square

Definition 7.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a \mathcal{Z}_p -summable nonunital semifinite spectral triple, relative to (\mathcal{N}, τ) with $\mathcal{A} \subseteq \mathcal{B}^2$, $\mathcal{B} \subseteq B_p$. We say $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smooth to order k if for all $b \in \mathcal{B}$ and for all $0 \leq j \leq k$, $\delta^j(b) \in B_p$. Here δ is the unbounded derivation given by $\delta(T) := [|\mathcal{D}|, T]$, $T \in \mathcal{N}$.*

Now we check that being smooth to order $[p] + 1$ ($[.]$ is the integer-part function) is enough to ensure that $b_1[b_2, (1 + \mathcal{D}^2)^{-p/2}] \in Z_1^0$ for $b_1, b_2 \in \mathcal{B} \subseteq B_p$.

Proposition 7.7. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable nonunital semifinite spectral triple with spectral dimension p , and which is smooth to order $[p] + 1$. Suppose moreover that $\mathcal{A} \subseteq \mathcal{B}^2$ for $\mathcal{B} \subseteq B_p$. Then for all $b_1, b_2 \in \mathcal{B} \subseteq B_\zeta$, $b_1[b_2, (1 + |\mathcal{D}|)^{-p}]$ is trace class.*

Proof. We may write

$$(1 + |\mathcal{D}|)^{-p} = \frac{1}{2\pi i} \int_\ell \lambda^{-p} (\lambda - 1 - |\mathcal{D}|)^{-1} d\lambda,$$

where $\ell = \{a + iv : v \in \mathbb{R}, a = 1/2\}$. Then we have

$$\begin{aligned} [(1 + |\mathcal{D}|)^{-p}, b_2] &= \frac{1}{2\pi i} \int_{\ell} \lambda^{-p} (\lambda - 1 - |\mathcal{D}|)^{-1} [|\mathcal{D}|, b_2] (\lambda - 1 - |\mathcal{D}|)^{-1} d\lambda \\ &= p(1 + |\mathcal{D}|)^{-p-1} [|\mathcal{D}|, b_2] - \frac{1}{2\pi i} \int_{\ell} \lambda^{-p} (\lambda - 1 - |\mathcal{D}|)^{-1} [(\lambda - 1 - |\mathcal{D}|)^{-1}, [|\mathcal{D}|, b_2]] d\lambda. \end{aligned}$$

Repeat the previous resolvent trick to obtain

$$[(1 + |\mathcal{D}|)^{-p}, b_2] = p(1 + |\mathcal{D}|)^{-p-1} [|\mathcal{D}|, b_2] - \frac{1}{2\pi i} \int_{\ell} \lambda^{-p} (\lambda - 1 - |\mathcal{D}|)^{-2} [|\mathcal{D}|, [|\mathcal{D}|, b_2]] (\lambda - 1 - |\mathcal{D}|)^{-1} d\lambda.$$

Now iterate the process and multiply on the left by b_1 to obtain

$$\begin{aligned} b_1[(1 + |\mathcal{D}|)^{-p}, b_2] &= p b_1(1 + |\mathcal{D}|)^{-p-1} [|\mathcal{D}|, b_2] - p(p+1) b_1(1 + |\mathcal{D}|)^{-p-2} [|\mathcal{D}|, [|\mathcal{D}|, b_2]] \dots \\ &\quad \dots - \frac{1}{2\pi i} \int_{\ell} \lambda^{-p} b_1 (\lambda - 1 - |\mathcal{D}|)^{-[p]-1} [|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]] \dots] (\lambda - 1 - |\mathcal{D}|)^{-1} d\lambda, \end{aligned}$$

where there are $[p] + 1$ commutators in the integrand. Under the smoothness assumption $\delta^k(b_2) \in B_p$ for $k \leq [p] + 1$ we may use Lemma 4.7 to argue that for each $\lambda \in \ell$ the product

$$b_1 (\lambda - 1 - |\mathcal{D}|)^{-[p]-1} [|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]] \dots]$$

is trace class because $b_1(1 + |\mathcal{D}|)^{-[p]-1} [|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]] \dots]$ is trace class. Simple estimates now show that the integral converges in trace norm. The result follows. \square

Example. Take $(\mathcal{N}, \tau) = (\mathcal{B}(L^2(\mathbb{R}^p, S)), \text{Tr})$, where S is the trivial spinor bundle, and Tr the usual operator trace. We let \not{D} be the standard Dirac operator, $G = (1 + \not{D}^2)^{-p/2}$, $\mathcal{H} = L^2(\mathbb{R}^p, S)$, and \mathcal{A} the bounded smooth integrable functions all of whose partial derivatives are bounded and integrable. Let $b \in L^2(\mathbb{R}^p)$ be the function $b(x) = (1 + \|x\|^2)^{-p/4-\epsilon/2}$. Then $\mathcal{A} \ni a = b^2$ satisfies the hypotheses of Corollary 7.4 and

$$\begin{aligned} \text{Tr}_{\omega} ((1 + \|x\|^2)^{-p/2-\epsilon} G) &= \tilde{\omega} - \lim_{r \rightarrow \infty} \frac{1}{r} \text{Tr} \left((1 + \|x\|^2)^{-p/4-\epsilon/2} G^{1+\frac{1}{r}} (1 + \|x\|^2)^{-p/4-\epsilon/2} \right) \\ &= C_p \int_{\mathbb{R}^p} (1 + \|x\|^2)^{-p/2-\epsilon} d^p x, \end{aligned}$$

the final line following from an explicit calculation similar to that in [33, Corollary 14].

Recall that an algebra \mathcal{A}_c has local units when, for any finite subset of elements $\{a_1, \dots, a_k\}$ of \mathcal{A}_c , there exists $u \in \mathcal{A}_c$ such that $ua_i = a_i u = a_i$ for $i = 1, \dots, k$. In the example above we could take \mathcal{A}_c to be the smooth compactly supported functions, and apply the theory in [33] to elements of \mathcal{A}_c . However, the local units based theories of [18, 33] can not accomodate the integration of the function b^2 . Indeed, since the function b^2 is nowhere vanishing, the condition $b^2 u = b^2$ implies necessarily that u is the constant unit function, which does not belong to \mathcal{A} . Thus even in the classical case of manifolds our approach to integration allows the integration of many more functions. Similar examples may be constructed on any complete manifold, [33].

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